Solutions for Homework Assignment #4

Answer to Question 1.

a. Special4Sat= \( \mathcal{L}(M) \) for the following polynomial-time nondeterministic machine \( M \).

On input \( \alpha \), \( M \) checks that \( \alpha = \langle F \rangle \) where \( F \) is a 4CNF formula (and if not rejects).

\( M \) then guesses a truth assignment, and computes if that assignment makes at least two literals in every clause true; if so, \( M \) accepts; if not, \( M \) rejects.

b. Let \( f : \{0,1\}^* \rightarrow \{0,1\}^* \) be the function computed by the following polynomial-time algorithm.

On input \( \alpha \), we first check that \( \alpha = \langle F \rangle \) where \( F = (L_{1,1} \lor L_{1,2} \lor V_{1,3}) \land (L_{2,1} \lor L_{2,2} \lor L_{2,3}) \land \ldots \land (L_{m,1} \lor L_{m,2} \lor L_{m,3}) \) is a 3CNF formula (and if not output something not in Special4Sat).

Let \( x \) be a variable not appearing in \( F \). We then compute \( f(\alpha) = F' \) where \( F' = (L_{1,1} \lor L_{1,2} \lor L_{1,3} \lor x) \land (L_{2,1} \lor L_{2,2} \lor L_{2,3} \lor x) \land \ldots \land (L_{m,1} \lor L_{m,2} \lor L_{m,3} \lor x) \). We leave it as an exercise to check that \( \langle F \rangle \in 3SAT \iff \langle F' \rangle \in \text{Special4Sat} \).

Answer to Question 2.

We assume we have a polynomial-time algorithm \( \text{ALG} \) for computing membership in SS. Say are given \( \langle a_1, a_2, \ldots, a_m, t \rangle \in \text{SS} \). We want to compute (in polynomial time) a set \( A \subseteq \{1, 2, \ldots, m\} \) such that \( \sum_{i \in A} a_i = t \).

For \( i = 1, 2, \ldots, m \) we will decide whether or not to put \( i \) into \( A \). Here is how we decide whether to put \( i \) into \( A \): we compute (using \( \text{ALG} \)) if \( \langle a_2, \ldots, a_m, t \rangle \in \text{SS} \); if so, we do not put \( i \) into \( A \), and we continue with input \( \langle a_2, \ldots, a_m, t \rangle \), if not, we put \( i \) into \( A \), and continue with input \( \langle a_2, \ldots, a_m, t - a_1 \rangle \). We continue in this way for \( i = 2, \ldots, m \).

Answer to Question 3.

a. The following nondeterministic polynomial-time algorithm accepts-half-clique:

Given an input string, first check that it is of the form \( \langle G \rangle \) such that \( G = (V, E) \) is an undirected graph, and if not reject.

Assuming \( G = (V, E) \), guess a set \( C \subseteq V \); check that there is an edge between each pair of vertices in \( C \) (there are \( \leq |V|^2 \) such pairs) and that \( |C| \geq |V|/2 \); if so, accept; if not reject.

b. We want a polynomial-time function \( f : \{0,1\}^* \rightarrow \{0,1\}^* \) such that for all \( \alpha \in \{0,1\}^*, \alpha \in \text{CLIQUE} \iff f(\alpha) \in \text{HALF-CLIQUE} \). We define \( f \) as follows.

Given \( \alpha \), first check that \( \alpha = \langle G, k \rangle \) where \( G = (V, E) \) is an undirected graph and \( k \) is a nonnegative integer \( \leq |V| \); if not, let \( f(\alpha) \) be anything not in \( \text{HALF-CLIQUE} \). So now assume \( \alpha = \langle G, k \rangle \), \( G = (V, E) \) an undirected graph, \( 0 \leq k \leq |V| \). We wish to construct \( f(\langle G, k \rangle) = \langle G' \rangle \) for a graph \( G' = (V', E') \) such that

\[
G \text{ has a clique of size } k \iff G' \text{ has a clique of size } \geq |V'|/2
\]

The idea is that if \( k \geq |V|/2 \), we form \( G' \) by adding \( 2k - |V| \) new, totally disconnected vertices to \( G \); if \( k \leq |V|/2 \), we form \( G' \) by adding a clique of \( |V| - 2k \) of new vertices that have edges to all the old vertices. More specifically:
CASE: $k \geq |V|/2$.
Note that $2k - |V| \geq 0$. Let $V_0$ be a set of $2k - |V|$ vertices none of which are in $V$. Let $G' = (V', E)$ where $V' = V \cup V_0$. Note that $|V'| = 2k$, and so $|V'|/2 = k$. It is easy to see (exercise) that $G$ has a clique of size $k \iff G'$ has a clique of size $k$.

CASE: $k < |V|/2$.
Note that $|V| - 2k \geq 0$. Let $V_0$ be a set of $|V| - 2k$ vertices none of which are in $V$. Let $G' = (V', E')$ where $V' = V \cup V_0$, and where $E'$ consists of all of $E$, as well as all the edges between every vertex of $V_0$ and every other vertex of $V_0 \cup V = V'$. Note that $|V'| = 2|V| - 2k$, and so $|V'|/2 = |V| - k$.
Consider a largest possible clique $C'$ in $G'$. $C'$ will equal $V_0 \cup C$ where $C \subseteq V$ is a largest possible clique of $G$ (do you see why?). $|C'| = |C| + (|V| - 2k)$, so $|C'| \geq |V'|/2 = |V| - k \iff |C| \geq k$. So $G$ has a clique of size $k \iff G'$ has a clique of size $|V| - k = |V'|/2$. 