Answer to Question 1.

a. The following algorithm accepts precisely \( L \).

If the input is not of the form \(<M>\) where \( M \) is a Turing machine over \{0, 1\}, then reject.

So say that the input is \(<M>\) where \( M \) is a Turing machine over \{0, 1\}. We then dovetail as follows: run \( M \) on all strings of length \( \leq 1 \) for 1 step, then run \( M \) on all strings of length \( \leq 2 \) for 2 steps, then run \( M \) on all strings of length \( \leq 3 \) for 3 steps, \ldots. If in any phase, two strings of the same length are found such that \( M \) is found to halt on both of them, and accept one and reject the other, then halt and accept.

b. We will show that \( B_{TM} \leq_m L \). We will exhibit a computable \( f : \{0, 1\}^* \to \{0, 1\}^* \) such that for all \( x \in \{0, 1\}^* \), \( x \in B_{TM} \iff f(x) \in L \). We compute \( f \) as follows.

Let \( x \in \{0, 1\}^* \). If \( x \) is not of the form \(<M>\) where \( M \) is a Turing machine over \{0, 1\}, then let \( f(x) \) be anything not in \( L \). So say \( x \) is of the form \(<M>\) where \( M \) is a Turing machine over \{0, 1\}. We then let \( f(x) = <M'> \) where \( M' \) is the following Turing machine over \{0, 1\}:

- If the input is the string 0, then reject; otherwise, erase the input and run \( M \) on \( \epsilon \), accepting/rejecting as \( M \) does.

We want to show that \( x \in B_{TM} \iff f(x) \in L \).

To show \( \Rightarrow \), assume that \( M \) accepts \( \epsilon \). Then \( M' \) rejects 0 and accepts 1, so \(<M'> \in L \).

To show \( \Leftarrow \), assume \( M \) does not accept \( \epsilon \). Then \( M' \) does not accept anything, and so \(<M'> \notin L \).

Answer to Question 2. We will disprove the assertion, by presenting a counterexample.

Let \( L_1, L, L_2 \) be the following languages over \{0, 1\}:

\[
\begin{align*}
L_1 &= \emptyset \\
L &= A_{tm} \\
L_2 &= \{0, 1\}^*
\end{align*}
\]

\( L_1 \subseteq L \subseteq L_2 \), and \( L_1 \) and \( L_2 \) are decidable, but \( L \) is not decidable.

Answer to Question 3.

a. Define the computable function \( f : \{0, 1\}^* \to \{0, 1\}^* \) as follows. On input \( x \in \{0, 1\}^* \) compute \( f(x) \) as follows:

- Check that \( x \) is of the form \( \langle M \rangle \) where \( M \) is a Turing machine over \{0, 1\}, and if not, reject.
- So assume \( x \) is \( \langle M \rangle \) where \( M \) is a Turing machine over \{0, 1\}.
- Compute the Turing machine \( M' \) which on every input, ignores its input and runs \( M \) on the blank tape, accepting or rejecting whenever \( M \) does. Let \( f(x) = \langle M' \rangle \).
- If \( \epsilon \in L(M) \), then \( M' \) accepts everything, and so \( \langle M' \rangle \in L \).
- If \( \epsilon \notin L(M) \), then \( M' \) accepts nothing, and so \( \langle M' \rangle \notin L \).
- So \( x \in B_{TM} \iff f(x) \in L \).
b. Define the computable function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ as follows. On input $x \in \{0, 1\}^*$ compute $f(x)$ as follows:

Check that $x$ is of the form $\langle M \rangle$ where $M$ is a Turing machine over $\{0, 1\}$, and if not, let $f(x)$ be anything in $L$.

So assume $x$ is $\langle M \rangle$ where $M$ is a Turing machine over $\{0, 1\}$.

Compute the Turing machine $M'$ which on input $\alpha$, runs $M$ on the blank tape for $|\alpha|$ steps; if $M$ accepts within $|\alpha|$ steps, then $M'$ rejects, otherwise $M'$ accepts. Let $f(x) = \langle M' \rangle$.

If $M$ accepts $\epsilon$, say in $k$ steps, then $M'$ rejects everything of length $\geq k$, and so $\langle M' \rangle \notin L$.

If $\epsilon \notin L(M)$, then $M'$ accepts everything, and so $\langle M' \rangle \in L$.

So $x \in B_{TM} \Leftrightarrow f(x) \in \overline{L}$.

**Answer to Question 4.** Say that $L = L(M_0)$ and $L$ satisfies condition 1; we want to show that $L$ does not satisfy condition 2. Let $L'$ be as defined in the Hint.

We first show that $L'$ is decidable. In fact, the following algorithm decides membership in $L'$:

Given $\alpha$, first check that $\alpha$ is of the form $\langle M, i \rangle$ where $M$ is a Turing machine over $\{0, 1\}$ and $i$ is a positive integer (and if not reject). Next, run $M_0$ on $\langle M \rangle$ for (up to) $i$ steps and if $M_0$ does not accept within that time, then reject. So assume that $M_0$ accepts $M$ within $i$ steps; by 1, this means that $M$ is a deciding machine. So run $M$ on $\langle M, i \rangle$ until it halts; if $M$ accepts, then reject; if $M$ rejects, then accept.

Since $L'$ is decidable, it is sufficient to show that there is no Turing machine $M'$ over $\{0, 1\}$ such that $\langle M' \rangle \in L$ and $L(M') = L'$. So consider any $\langle M' \rangle \in L$. we will show that $L(M') \neq L'$.

Since $\langle M' \rangle \in L = L(M_0)$, we know that $M_0$ accepts $\langle M' \rangle$, say in $i$ steps.

Now consider whether or not $\langle M', i \rangle \in L'$. Since $M_0$ accepts $\langle M' \rangle$ within $i$ steps, we see from the definition of $L'$ that

$$\langle M', i \rangle \in L' \Leftrightarrow \langle M', i \rangle \notin L(M')$$

So $L(M') \neq L'$. 