Some Examples of Mapping Reductions

Here are two examples of mapping reductions from the February 4 tutorial.
Let $B_{TM} = \{ <M> \mid M \text{ is a Turing machine over } \{0,1\} \text{ and } M \text{ accepts } \epsilon \} \subseteq \{0,1\}^*$. Since $B_{TM}$ is recognizable but not decidable, we know that $\overline{B_{TM}}$ is not recognizable.

Let $ALL_{TM} = \{ <M> \mid M \text{ is a Turing machine over } \{0,1\} \text{ and } M \text{ accepts every input} \} \subseteq \{0,1\}^*$. We wish to show that both $ALL_{TM}$ and $\overline{ALL_{TM}}$ are not recognizable.

I. To show that $\overline{ALL_{TM}}$ is not recognizable, it is sufficient to show that $B_{TM} \leq_m ALL_{TM}$, or equivalently,

$$B_{TM} \leq_m ALL_{TM}$$

To this end, we define the computable $f : \{0,1\}^* \rightarrow \{0,1\}^*$ as follows:
Let $\alpha \in \{0,1\}^*$. We first check that $\alpha$ is of the form $<M>$ where $M$ is a Turing machine over $\{0,1\}$ (and if not, we let $f(\alpha)$ be anything not in $ALL_{TM}$). We then compute $f(\alpha) = <M'>$ where $M'$ is the Turing machine over $\{0,1\}$ that on every input, ignores its input and runs $M$ on $\epsilon$, accepting or rejecting if and when $M$ does.

If $M$ accepts $\epsilon$, then $M'$ accepts every input. If $M$ does not accept $\epsilon$, then it is not the case that $M'$ accepts every input (in fact, $M'$ accepts no input). So

$$\epsilon \in L(M) \Leftrightarrow L(M') = \{0,1\}^*$$

$$\alpha \in B_{TM} \Leftrightarrow f(\alpha) \in ALL_{TM}$$

II. This direction is tricky.
To show that $ALL_{TM}$ is not recognizable, it is sufficient to show that $\overline{B_{TM}} \leq_m ALL_{TM}$, or equivalently,

$$B_{TM} \leq_m \overline{ALL_{TM}}$$

To this end, we define the computable $f : \{0,1\}^* \rightarrow \{0,1\}^*$ as follows:
Let $\alpha \in \{0,1\}^*$. We first check that $\alpha$ is of the form $<M>$ where $M$ is a Turing machine over $\{0,1\}$ (and if not, we let $f(\alpha)$ be anything in $ALL_{TM}$). We then compute $f(\alpha) = <M'>$ where $M'$ is the Turing machine over $\{0,1\}$ that behaves as follows: on input $x$, $M'$ runs $M$ on $\epsilon$ for $|x|$ steps; if $M$ accepts within that time, then $M'$ halts and rejects $x$; if $M$ does not accept within that time, then $M'$ halts and accepts $x$.

If $M$ accepts $\epsilon$, say within $k$ steps, then $M'$ rejects every input of length $\geq k$, and so it is not the case that $M'$ accepts every input. If $M$ does not accept $\epsilon$, then $M'$ accepts every input. So

$$\epsilon \in L(M) \Leftrightarrow L(M') \neq \{0,1\}^*$$

$$\alpha \in B_{TM} \Leftrightarrow f(\alpha) \in \overline{ALL_{TM}}$$