Question 1 [Greedy Algorithm]

Given a nonempty set $S$ of $n$ elements. A nonempty family $L$ of subsets of $S$ is called nice if it satisfies the following conditions:

1. **Inclusion property**: For every subsets $A, B \in S$, if $A \subseteq B$ and $B \in L$ then $A \in L$. (In other words, if $B$ is a member of $L$ then all subsets of $B$ are also members of $L$. Note that the empty set $\emptyset$ is necessarily a member of $L$.)

2. **Exchange condition**: If $A \in L$ and $B \in L$ and $|A| < |B|$ (here $|A|$, $|B|$ denotes the number of elements in $A$ and $B$ respectively), then there is some element $x \in B - A$ such that $A \cup \{x\} \in L$.

A subset $A$ in $L$ is called a top set if there is no other set $B$ in $L$ such that $A \subset B$.

Given a set $S = \{1, 2, \ldots, n\}$ where each element $i$ has a weight $w(i)$, and a nice family $L$ of subsets of $S$. The weight of a set $A$ in $L$ is the total weight of the elements in $A$:

$$w(A) = \sum_{i \in A} w(i)$$

The problem is to find a set $A$ in $L$ with maximum weight. Notice that any set $A$ in $L$ of maximum weight must be a top set. Also, $L$ might have as many as $2^n$ members. Therefore going through every member of $L$ is not an option here, because the problem can be solved in polynomial time using the greedy approach.

**(a)** Give a Greedy algorithm that finds a maximum weight subset in $L$. Prove that your algorithm is correct.

**(b)** To analyze the running time of your algorithm, we assume that checking whether a subset $A \subset S$ is a member of $L$ takes time $t(n)$. (This checking time only depends on $n$—the number of elements in $S$—and not on the set $A$.) What is the running time of your algorithm in terms of $n$ and $t(n)$ (state your answer using $O$ notation)?

Question 2 [Greedy Algorithm]

Consider the problem that, given a set $\{x_1, \ldots, x_n\}$ of points on the real line, determines the smallest set of unit-length closed intervals that contains all of the given points.

For example, on input $\{-2.7, 1.5, 2.5\}$, the following set of two unit intervals is an optimal solution:

$$\{[-3, -2], [1.5, 2.5]\}$$

**(a)** Give the pseudo-code for a greedy algorithm that solves the above problem and prove that your algorithm is correct.
b) What is the running time of your algorithm?

Question 3 [Approximation Algorithm]