

1. Recall that a Hamiltonian cycle in a graph $G = (V, E)$ is a cycle that visits each vertex exactly once. In other words, if $n = |V|$, a Hamiltonian cycle is a permutation $\varphi(1), \dots, \varphi(n)$ of V such that $(\varphi(i), \varphi(i+1)) \in E$ for $1 \leq i < n$ and $(\varphi(n), \varphi(1)) \in E$. The decision problem HAM-CYCLE is described below.

Instance: A directed graph $G = (V, E)$

Question: Does G have a Hamiltonian cycle?

Prove that HAM-CYCLE is **NP**-complete.

Solution: It is easy to see that HAM-CYCLE is in **NP**: the non-deterministic algorithm guesses a permutation of the vertices and checks that it is a cycle in G . To prove that HAM-CYCLE is **NP**-hard, we'll show $\text{HAM-PATH} \leq_p \text{HAM-CYCLE}$. Let $\langle G, s, t \rangle$ be an instance of HAM-PATH, where $G = (V, E)$ and $s, t \in V$. The reduction maps $\langle G, s, t \rangle$ to $\langle G' \rangle$, where $G' = (V', E')$ and

$$\begin{aligned} V' &= V \cup \{s^*, t^*\} \\ E' &= E \cup \{(s^*, s), (t, t^*), (t^*, s^*)\} \end{aligned}$$

We need to prove that G has a Hamiltonian path from s to t iff G' has a Hamiltonian cycle. Assume first that G has a Hamiltonian path π from s to t . Then there is a Hamiltonian cycle in G' that visits s^* , then s , then follows π to t , then visits t^* and returns to s^* . Now assume that G' has a Hamiltonian cycle. Since s^* has only one out-neighbour s , and only one in-neighbour t^* , and since t^* has only one in-neighbour t , the Hamiltonian cycle must visit t, t^*, s^*, s in that order. The rest of the cycle is therefore a path from s to t that visits every vertex in $V' \setminus \{s^*, t^*\} = V$, i.e. it is a Hamiltonian path from s to t in G .

2. The traveling salesman decision problem (TSP) is defined below.

Instance: A distance function $d : [n]^2 \mapsto \mathbb{R}^+$, and $D \in \mathbb{R}^+$

Question: Does there exist a permutation φ of $\{1, \dots, n\}$ such that

$$\sum_{i=1}^{n-1} d(\varphi(i), \varphi(i+1)) + d(\varphi(n), \varphi(1)) \leq D$$

Prove that TSP is **NP**-complete.

Solution: First, show $\text{TSP} \in \text{NP}$: The non-deterministic algorithm guesses a permutation and checks that the total distance is D or less. Now to prove that TSP is **NP**-hard, we'll show $\text{HAM-CYCLE} \leq_p \text{TSP}$. Let $G = (V, E)$ be an instance of HAM-CYCLE, and assume $V = \{1, \dots, n\}$. The reduction maps $\langle G \rangle$ to $\langle d, D \rangle$, where $d : [n]^2 \mapsto \mathbb{R}^+$ is defined by

$$d(i, j) = \begin{cases} 1, & \text{if } (i, j) \in E \\ n+1, & \text{otherwise} \end{cases}$$

and $D = n$. Now we need to prove that $\langle G \rangle \in \text{HAM-CYCLE}$ iff $\langle d, D \rangle \in \text{TSP}$. Assume first that G has a Hamiltonian cycle, i.e. a permutation $\varphi(1), \dots, \varphi(n)$ of V such that $(\varphi(i), \varphi(i+1)) \in E$ for $1 \leq i < n$ and $(\varphi(n), \varphi(1)) \in E$. Then by construction we have $d(\varphi(i), \varphi(i+1)) = 1$ for $1 \leq i < n$ and $d(\varphi(n), \varphi(1)) = 1$, so

$$\sum_{i=1}^{n-1} d(\varphi(i), \varphi(i+1)) + d(\varphi(n), \varphi(1)) = n = D$$

Thus if G has a Hamiltonian cycle, then d has a TSP tour with total distance D . Now assume that there is a TSP tour in d with total distance D , i.e. a permutation $\varphi(1), \dots, \varphi(n)$ such that

$$\sum_{i=1}^{n-1} d(\varphi(i), \varphi(i+1)) + d(\varphi(n), \varphi(1)) \leq D = n$$

Since $d(i, j) \geq 1$ for all i and j , and there are n terms in the sum, it follows that $d(\varphi(i), \varphi(i+1)) = 1$ for $1 \leq i < n$ and $d(\varphi(n), \varphi(1)) = 1$. By construction, this implies that $(\varphi(i), \varphi(i+1)) \in E$ for $1 \leq i < n$ and $(\varphi(n), \varphi(1)) \in E$, i.e. $\varphi(1), \dots, \varphi(n)$ is a Hamiltonian cycle in G .