

1. Show that every infinite language contains an unrecognizable subset.

**Solution:** Let  $L$  be an infinite language, and let  $s_1, s_2, s_3, \dots$  be the strings in  $L$ . It is enough to show that the subsets of  $L$  are uncountable: since the recognizable languages are countable, this implies that there are subsets of  $L$  that are not recognizable.

To prove that the subsets of  $L$  are uncountable, assume for a contradiction that there exists a function  $\phi$  mapping  $\mathbb{N}$  onto the subsets of  $L$ . Define the set  $S \subseteq L$  as follows: for each  $s_i \in L$ ,  $s_i \in S$  iff  $s_i \notin \phi(i)$ . Then  $S$  is a subset of  $L$ , but for all  $i$ ,  $S \neq \phi(i)$ , contradicting the assumption that  $\phi$  is onto.

2. Show that if  $D$  is decidable and  $L$  is unrecognizable, and  $D \cap L = \emptyset$ , then  $D \cup L$  is unrecognizable.

**Solution:** Let  $D$  be decidable and let  $L$  be unrecognizable, such that  $D \cap L = \emptyset$ , and suppose for a contradiction that  $D \cup L$  is recognizable. Let  $M_1$  be a TM that decides  $D$ , and let  $M_2$  be a TM that recognizes  $D \cup L$ . We will construct a TM  $M_L$  that recognizes  $L$ , contradicting the hypothesis. Define  $M_L =$  "on input  $w$ ...

1. Run  $M_1$  on input  $w$
2. Run  $M_2$  on input  $w$
3. If  $M_1$  rejects and  $M_2$  accepts then accept; else reject

We need to show that  $M_L$  accepts  $w$  iff  $w \in L$ . Suppose  $w \in L$ . Then  $w \in D \cup L$  but  $w \notin D$  (since  $D \cap L = \emptyset$ ), so  $M_2$  accepts and  $M_1$  rejects, causing  $M_L$  to accept  $w$ . On the other hand, if  $w \notin L$  then either  $w \in D$  or  $w \notin D \cup L$ ; in the first case  $M_1$  accepts and in the second case  $M_2$  does not accept, so in both cases  $M$  does not accept  $w$ .

3. Show that there exists a decidable language  $L$  and unrecognizable languages  $S_1, S_2$  such that  $S_1 \subseteq L \subseteq S_2$ .

**Solution:** Let  $L$  be any decidable language such that both  $L$  and  $\bar{L}$  are infinite. For example,  $L = \{0^n \mid n \in \mathbb{N}\}$ . By exercise 1,  $L$  has an unrecognizable subset  $S_1$ , and  $\bar{L}$  has an unrecognizable subset  $S$ . As  $S$  is a subset of  $\bar{L}$  it follows that  $S \cap L = \emptyset$ , so exercise  $w$  implies that  $S_2 = S \cup L$  is unrecognizable. So we have that  $S_1$  and  $S_2$  are unrecognizable and  $S_1 \subseteq L \subseteq S_2$ .