1. Give a low-level description of a Turing machine that decides the following language.

$$L_1 = \{x\#y \mid x, y \in \{0,1\}^*, x = y\}$$

Solution: The idea is to compare the first symbols, then the second, and so on. The trick to keeping track of which symbols are the "current" ones is to use new symbols $\underline{0}$ and $\underline{1}$. Initially, the first symbol of the input should be underlined (unless it is #), and the first symbol after the # is underlined (unless it is $\sqcup$). The underlined symbols are compared, and then each underline is moved one space to the right.

Formally, the TM has input alphabet $\Sigma = \{0, 1, \#\}$ and tape alphabet $\Gamma = \{0, 1, \#, \underline{0}, \underline{1}, \sqcup\}$. For the definition of the states and the transition function, we'll describe the operation in terms of several "phases". At a high level:

- Phase 1: Check that there is exactly one # symbol
- Phase 2: Check whether x, y are empty
  - If both are empty, accept
  - If only one is empty, reject
- Phase 3: Underline first symbol of x and first symbol of y
- Phase 4: Compare underlined symbols
  - If different, reject
- Phase 5: Move underlines one space to the right
  - If x ends but not y, or y ends but not x, reject
  - If both end, accept
  - Goto Phase 4

Phase 1

In this phase, the TM checks that the input has the form $x\#y$ with $x, y \in \{0,1\}^*$. Essentially it skips over any number of 0's and 1's until a # is found, then makes sure there are no more #'s after that point.

- q_0: moving right to first # (if none found, reject)
- q_1: make sure no more #'s occur
- q_2: rewind to beginning of input

State	Symbol	State	Symbol	Direction
q_0	$0,1$	q_0	$0,1$	R
q_0	$\#$	q_1	$\#$	R
q_0	$\sqcup$	q_{reject}	$*$	$*$
q_1	$0,1$	q_1	$0,1$	R
q_1	$\#$	q_{reject}	$*$	$*$
q_1	$\sqcup$	q_2	$\sqcup$	L
q_2	$0,1,\#$	q_2	$0,1,\#$	L
q_2	$\sqcup$	q_3	$\sqcup$	R

Phase 2

Next, the TM checks whether x and y are empty. If they are both empty, it accepts, and if only one is empty, it rejects. If both are non-empty it goes on to the next phase.

- q_3: check if x is empty (i.e. if first symbol is #)
- q_4: x is empty, check if y is empty (i.e. current symbol is $\sqcup$)
- q_5: x is not empty, move to beginning of y
- q_6: x is not empty, check if y is empty (i.e. current symbol is $\sqcup$)
- q_7: both sides non-empty (rewind)

State	Symbol	State	Symbol	Direction
q_3	$0, 1$	q_5	$0, 1$	R
q_3	#	q_4	#	R
q_4	$0, 1$	q_{reject}	$*$	$*$
q_4	$\sqcup$	q_{accept}	$*$	$*$
q_5	$0, 1$	q_5	$0, 1$	R
q_5	#	q_6	#	R
q_6	$0, 1$	q_7	$0, 1$	L
q_6	$\sqcup$	q_{reject}	$*$	$*$
q_7	$0, 1, \#$	q_7	$0, 1, \#$	L
q_7	$\sqcup$	q_8	$\sqcup$	R

Phase 3

At this point we know that we are at the beginning of the input, that the input has the form $x\#y$ with $x, y \in \{0,1\}^*$, and that x and y are both non-empty. The next stage is to underline the first symbols of x and of y.

- q_8: underline first symbol of x
- q_9: move right past #
- q_{10}: underline first symbol of y
- q_{11}: rewind

State	Symbol	State	Symbol	Direction
q_8	0	q_9	$\underline{0}$	R
q_8	1	q_9	$\underline{1}$	R
q_9	$0,1$	q_9	$0,1$	R
q_9	$\#$	q_{10}	$\#$	R
q_{10}	0	q_{11}	$\underline{0}$	L
q_{10}	1	q_{11}	$\underline{1}$	L
q_{11}	$0,1,\underline{0},\underline{1},\#$	q_{11}	$0,1,\underline{0},\underline{1},\#$	L
q_{11}	$\sqcup$	q_{12}	$\sqcup$	R

Phase 4

This phase compares the underlined symbols, and rejects if they are different.

- q_{12}: move right until an underlined symbol is found
- q_{13}: underlined symbol is $\underline{0}$, move to next underlined symbol, compare
- q_{14}: underlined symbol is $\underline{1}$, move to next underlined symbol, compare
- q_{15}: rewind

State	Symbol	State	Symbol	Direction
q_{12}	$0, 1$	q_{12}	$0, 1$	R
q_{12}	$\underline{0}$	q_{13}	$\underline{0}$	R
q_{12}	$\underline{1}$	q_{14}	$\underline{1}$	R
q_{13}	$0, 1, \#$	q_{13}	$0, 1, \#$	R
q_{13}	$\underline{0}$	q_{15}	$\underline{0}$	L
q_{13}	$\underline{1}$	q_{reject}	$*$	$*$
q_{14}	$0, 1, \#$	q_{14}	$0, 1, \#$	R
q_{14}	$\underline{0}$	q_{reject}	$*$	$*$
q_{14}	$\underline{1}$	q_{15}	$\underline{1}$	L
q_{15}	$0, 1, \underline{0}, \underline{1}, \#$	q_{15}	$0, 1, \underline{0}, \underline{1}, \#$	L
q_{15}	$\sqcup$	q_{16}	$\sqcup$	R

Phase 5

The final stage is to move the underlines one space to the right.

- q_{16}: move right until an underlined symbol found, remove underline
- q_{17}: underline current symbol if it is not # (in this case x is finished)
- q_{18}: x not finished, move right to underlined symbol, remove underline
- q_{19}: x is finished, move right to past underlined symbol
- q_{20}: underline current symbol if not $\sqcup$, else reject
- q_{21}: accept if current symbol is $\sqcup$, reject otherwise
- q_{22}: rewind and go back to state q_{12}

State	Symbol	State	Symbol	Direction
q_{16}	$0, 1$	q_{16}	$0, 1$	R
q_{16}	$\underline{0}$	q_{17}	0	R
q_{16}	$\underline{1}$	q_{17}	1	R
q_{17}	0	q_{18}	$\underline{0}$	R
q_{17}	1	q_{18}	$\underline{1}$	R
q_{17}	$\#$	q_{19}	$\#$	R
q_{18}	$0, 1, \#$	q_{18}	$0, 1, \#$	R
q_{18}	$\underline{0}$	q_{20}	0	R
q_{18}	$\underline{1}$	q_{20}	1	R
q_{19}	$0, 1, \#$	q_{19}	$0, 1, \#$	R
q_{19}	$\underline{0}, \underline{1}$	q_{21}	$\underline{0}, \underline{1}$	R
q_{20}	0	q_{22}	$\underline{0}$	L
q_{20}	1	q_{22}	$\underline{1}$	L
q_{20}	$\sqcup$	q_{reject}	$*$	$*$
q_{21}	$0, 1$	q_{reject}	$*$	$*$
q_{21}	$\sqcup$	q_{accept}	$*$	$*$
q_{22}	$0, 1, \underline{0}, \underline{1}, \#$	q_{22}	$0, 1, \underline{0}, \underline{1}, \#$	L
q_{22}	$\sqcup$	q_{12}	$\sqcup$	R

2. Give a low-level description of a Turing machine that decides the following language.

$$L_2 = \{x\#y \mid x, y \in \{0,1\}^*, y = \overline{x}\}$$

where $\overline{x}$ denotes the bitwise complement of x.

Solution: Flip the bits of x and then run the TM from the previous question.

- p_0: flip bits until # is found
- p_1: rewind and go to state q_0

State	Symbol	State	Symbol	Direction
p_0	0	p_0	1	R
p_0	1	p_0	0	R
p_0	#	p_1	#	L
p_1	$0,1$	p_1	$0,1$	L
p_1	$\sqcup$	q_0	$\sqcup$	R