

Part I (6 marks)

1. Suppose that A and B are languages, and that $A \leq_p B$. Circle one answer for each statement below.

If $A \in \mathbf{P}$ then $B \in \mathbf{P}$.

TRUE

 FALSE

I DON'T KNOW

If $B \in \mathbf{P}$ then $A \in \mathbf{P}$.

 TRUE

FALSE

I DON'T KNOW

If A is **NP**-hard then B is **NP**-hard.

 TRUE

FALSE

I DON'T KNOW

If B is **NP**-hard then A is **NP**-hard.

TRUE

 FALSE

I DON'T KNOW

2. Write down the formal definition of $A \leq_p B$.

Solution: There exists a function $f : \Sigma^* \mapsto \Sigma^*$ that is computable in polynomial time, such that for all $x \in \Sigma^*$:

$$x \in A \iff f(x) \in B$$

Student number:

Part II (10 marks)

A simple path in a graph is a path that does not contain any repeated vertices. The length of a path is the number of edges in the path. The decision problem LONGEST-PATH is defined below.

Instance: A directed graph $G = (V, E)$, and $k \in \mathbb{N}$

Question: Does G have a simple path of length k or longer?

Prove that LONGEST-PATH is **NP**-hard (note: you do not need to prove that it is in **NP**). Your answer will be marked on its structure as well as its content.

Solution: Show $\text{HAM-PATH-EXISTS} \leq_p \text{LONGEST-PATH}$. Note that a Hamiltonian path is a simple path of length $n - 1$, and conversely a simple path of length $n - 1$ is a Hamiltonian path. Define $f(\langle G \rangle) = \langle G, n - 1 \rangle$, where n is the number of vertices in G . Clearly f is computable in polynomial time: an algorithm for computing f simply counts the number n of vertices in the input graph and appends the number $n - 1$ to its input.

If $\langle G \rangle \in \text{HAM-PATH-EXISTS}$, then G has a Hamiltonian path, i.e. G has a simple path of length $n - 1$, so $f(\langle G \rangle) = \langle G, n - 1 \rangle \in \text{LONGEST-PATH}$.

Conversely, if $f(\langle G \rangle) = \langle G, n - 1 \rangle \in \text{LONGEST-PATH}$, then G has a simple path with $n - 1$ edges, and thus with n vertices. As a simple path with n vertices is a Hamiltonian path, it follows that $\langle G \rangle \in \text{HAM-PATH-EXISTS}$.

We have shown that f is computable in polynomial time, and for all graphs G , $\langle G \rangle \in \text{HAM-PATH-EXISTS} \iff f(\langle G \rangle) \in \text{LONGEST-PATH}$. Thus $\text{HAM-PATH-EXISTS} \leq_p \text{LONGEST-PATH}$. Since HAM-PATH-EXISTS is **NP**-hard it follows that LONGEST-PATH is **NP**-hard.

Student number:

Part III (14 marks)

A language is **co-NP**-complete if its complement is **NP**-complete.

A boolean formula φ is a tautology if $\varphi(\tau) = 1$ for every truth assignment τ . The decision problem TAUTOLOGY is:

Instance: A boolean formula φ

Question: Is φ a tautology?

Prove that TAUTOLOGY is **co-NP**-complete. Your answer will be marked on its structure as well as its content.

Solution: The complement of TAUTOLOGY is the language FALSIFIABLE, consisting of all boolean formulae φ for which there exists a falsifying assignment τ .

Instance: A boolean formula φ

Question: Does there exist a truth assignment τ such that $\varphi(\tau) = 0$?

To prove that TAUTOLOGY is **co-NP**-complete, we must prove that FALSIFIABLE is **NP**-complete, i.e. we must prove that it is in **NP**, and that it is **NP**-hard. Clearly FALSIFIABLE is in **NP**: the non-deterministic algorithm guesses a truth assignment τ and checks that $\varphi(\tau) = 0$. As evaluating a formula can be done in polynomial time, this algorithm runs in (non-deterministic) polynomial time.

To prove that FALSIFIABLE is **NP**-hard, we show $\text{SAT} \leq_p \text{FALSIFIABLE}$. Note that a formula φ is satisfiable iff its negation $\neg(\varphi)$ is falsifiable. The reduction is therefore defined as

$$f(\langle\varphi\rangle) = \langle\neg(\varphi)\rangle$$

Clearly f can be computed in polynomial time: an algorithm for computing f simply encloses its input in brackets and prepends a negation symbol \neg .

If $\langle\varphi\rangle \in \text{SAT}$ then there exists a truth assignment τ such that $\varphi(\tau) = 1$, and thus $\neg(\varphi)(\tau) = 0$, implying $\langle\neg(\varphi)\rangle = f(\langle\varphi\rangle) \in \text{FALSIFIABLE}$.

Conversely, if $\langle\neg(\varphi)\rangle = f(\langle\varphi\rangle) \in \text{FALSIFIABLE}$, then there exists a truth assignment τ such that $\neg(\varphi)(\tau) = 0$, and thus $\varphi(\tau) = 1$, implying $\langle\varphi\rangle \in \text{SAT}$.

We have shown that $\langle\varphi\rangle \in \text{SAT} \iff f(\langle\varphi\rangle) \in \text{FALSIFIABLE}$, and f is computable in polynomial time, so $\text{SAT} \leq_p \text{FALSIFIABLE}$. Since SAT is **NP**-hard it follows that FALSIFIABLE is **NP**-hard.

Student number:

Appendix: NP-complete problems

SAT

Instance: A boolean formula φ

Question: Is φ satisfiable?

3SAT

Instance: A boolean formula φ in 3CNF

Question: Is φ satisfiable?

SUBSET-SUM

Instance: $s_1, \dots, s_n, t \in \mathbb{Z}$

Question: Does there exist a subset $S \subseteq \{1, \dots, n\}$ such that $\sum_{i \in S} s_i = t$?

PARTITION

Instance: $s_1, \dots, s_n \in \mathbb{Z}$

Question: Does there exist $S \subseteq \{1, \dots, n\}$ such that $\sum_{i \in S} s_i = \sum_{j \notin S} s_j$?

KNAPSACK

Instance: $w_1, \dots, w_n \in \mathbb{N}; p_1, \dots, p_n \in \mathbb{N}; W, P \in \mathbb{N}$

Question: Does there exist $S \subseteq \{1, \dots, n\}$ such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} p_i \geq P$?

VERTEX-COVER

Instance: A graph $G = (V, E)$, and $k \in \mathbb{N}$

Question: Does G have a vertex cover of size k or smaller?

INDEPENDENT-SET

Instance: A graph $G = (V, E)$ and $k \in \mathbb{N}$

Question: Does G have an independent set of size k or larger?

CLIQUE

Instance: A graph $G = (V, E)$, and $k \in \mathbb{N}$

Question: Does G have a clique of size k or larger?

SET-COVER

Instance: Sets S_1, \dots, S_n , and $k \in \mathbb{N}$

Question: Does there exist $I \subseteq \{1, \dots, n\}$ such that $|I| \leq k$ and $\bigcup_{i \in I} S_i = \bigcup_{i=1}^n S_i$?

HAM-PATH

Instance: A directed graph $G = (V, E)$, and $s, t \in V$

Question: Does G have a Hamiltonian path from s to t ?

HAM-CYCLE

Instance: A directed graph $G = (V, E)$

Question: Does G have a Hamiltonian cycle?

HAM-PATH-EXISTS

Instance: A directed graph $G = (V, E)$

Question: Does G have a Hamiltonian path (between any two vertices)?

TSP

Instance: A distance function $d : \{1, \dots, n\} \times \{1, \dots, n\} \mapsto \mathbb{N}$, and $k \in \mathbb{N}$

Question: Does there exist a permutation $\varphi : \{1, \dots, n\} \mapsto \{1, \dots, n\}$ such that

$$\sum_{i=1}^{i-1} d(\varphi(i), \varphi(i+1)) + d(\varphi(n), \varphi(1)) \leq k$$