Test 1
(Sample solution)

First name: ______________________ Student number: |_|_|_|_|_|_|_|_|_|

Last name: ______________________ TA's name: ______________________

Tutorial room: ______________________

Read the following **carefully** before starting.

1. There are four pages to this test, including this one.
2. The duration of the test is 50 minutes. No aids are allowed.
3. Write your student number at the bottom of each page.
4. All answers must be written on this test booklet. You may use the back of the page for extra space. Scrap paper will be provided, but will not be marked.
5. If you state clearly that you do not know the answer to a question, you will receive 20% of the marks for that question.

Part I	______/10
Part II	______/10
Part III	______/10
Total	______/30

Part I (10 marks)

(a) Let L be a language. Circle one response for each statement below.

If L is decidable, then $\overline{L}$ is decidable.

[TRUE] FALSE DON'T KNOW

If L is recognizable, then $\overline{L}$ is recognizable.

TRUE [FALSE] DON'T KNOW

If L is recognizable but not decidable, then $\overline{L}$ is not recognizable

[TRUE] FALSE DON'T KNOW

If L is not recognizable, then $\overline{L}$ is not recognizable.

TRUE [FALSE] DON'T KNOW

If L is decidable and $L' \subseteq L$ then L' is decidable

TRUE [FALSE] DON'T KNOW

(b) For each language below, circle the **strongest** true statement.

$L_1 = \{\langle M, w \rangle \mid M \text{ accepts } w\}$

DECIDABLE [RECOGNIZABLE] UNRECOGNIZABLE DON'T KNOW

$L_2 = \{\langle M, w, t \rangle \mid M \text{ accepts } w \text{ within } t \text{ steps}\}$

[DECIDABLE] RECOGNIZABLE UNRECOGNIZABLE DON'T KNOW

$L_3 = \{\langle M, w \rangle \mid M \text{ does not accept } w\}$

DECIDABLE RECOGNIZABLE [UNRECOGNIZABLE] DON'T KNOW

Student number: ________

(c) Write down the formal definition of the statement $A \leq_m B$.

Solution:
There exists a computable function $f : \Sigma^* \mapsto \Sigma^*$ such that for all $x \in \Sigma^*$,

$$x \in A \iff f(x) \in B$$

Part II (10 marks)

Prove that the following language is recognizable.

$$L_4 = \{\langle M_1, M_2 \rangle \mid L(M_1) \cap L(M_2) \neq \emptyset\}$$

Note that your answer will be marked on its structure as well as its content.

Solution:
Define $M =$"on input $\langle M_1, M_2 \rangle$...

for $t \leftarrow 1, 2, 3, \ldots, \infty$ do
    for $i \leftarrow 1, 2, \ldots, t$ do
        run M_1 on s_i for t steps
        run M_2 on s_i for t steps
        if both accepted, then accept

If $\langle M_1, M_2 \rangle \in L_4$ then there exists $s_j \in L(M_1) \cap L(M_2)$, and so there exist T_1 and T_2 such that M_1 accepts s_j in T_1 steps, and M_2 accepts s_j in T_2 steps. Let $T = \max\{j, T_1, T_2\}$. Since each step of the outer loop is finite, if M has not already accepted beforehand, then it reaches $t = T$ and $i = j$, and accepts. On the other hand, if $\langle M_1, M_2 \rangle \notin L_4$ then $L(M_1) \cap L(M_2) = \emptyset$, so M will not find a string that is accepted by both machines and will therefore not accept. Thus $L(M) = L_4$.

Student number:

Part III (10 marks)

Prove that the following language is not recognizable.

$$L_5 = \{\langle M_1, M_2\rangle \mid L(M_1) \cap L(M_2) = \emptyset\}$$

Note that your answer will be marked on its structure as well as its content.

Solution:
Show EMPTY $\leq_p L_5$. Define $f(\langle M\rangle) = \langle M_1, M_2\rangle$, where $M_1 = M$ and M_2 is a Turing machine that always accepts. Then f is computable, since it simply appends the description of a particular, fixed Turing machine to its input. Also,

$$\begin{aligned}
\langle M\rangle \in \text{EMPTY} &\implies L(M) = \emptyset \\
&\implies L(M) \cap \Sigma^* = \emptyset \\
&\implies L(M_1) \cap L(M_2) = \emptyset \\
&\implies \langle M_1, M_2\rangle \in L_5
\end{aligned}$$

$$\begin{aligned}
\langle M_1, M_2\rangle \in L_5 &\implies L(M_1) \cap L(M_2) = \emptyset \\
&\implies L(M) \cap \Sigma^* = \emptyset \\
&\implies L(M) = \emptyset \\
&\implies \langle M\rangle \in \text{EMPTY}
\end{aligned}$$

So $\langle M\rangle \in \text{EMPTY} \iff f(\langle M\rangle) \in L_5$.

Student number: |_|_|_|_|_|_|_|_|_|