Partition

- The PARTITION decision problem is defined as:

  Instance: $p_1, \ldots, p_n \in \mathbb{Z}$
  Question: Does there exist $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} s_i = \sum_{j \notin S} s_j$?

- **Theorem:** PARTITION is **NP**-complete.

- Proof: Proving PARTITION $\in$ NP is an exercise. To prove that PARTITION is **NP**-hard, we'll show SUBSET-SUM $\leq_p$ PARTITION. The reduction function is: $f(\langle s_1, \ldots, s_n, t \rangle) = \langle p_1, \ldots, p_{n+1} \rangle$, where $p_i = s_i$ for $1 \leq i \leq n$, and

$$p_{n+1} = 2t - \sum_{i=1}^{n} s_i$$

  It's clear that the reduction operates in polynomial time: we only need to copy $s_1, \ldots, s_n$ to the output and add one new element p_{n+1}, and $t - \sum_{i=1}^{n} s_i$ is easily computed in polynomial time. We must argue that there exists $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} s_i = t$ iff there exists $S' \subseteq \{1, \ldots, n+1\}$ such that $\sum_{i \in S'} p_i = \sum_{j \notin S'} p_j$.

  ($\Rightarrow$) Suppose there exists $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} s_i = t$. Then $\sum_{i \in S} p_i = t$, and since $\sum_{i=1}^{n+1} p_i = 2t$, it follows that

$$\sum_{j \in \{1, \ldots, n+1\} \setminus S} p_i = t = \sum_{i \in S} p_i$$

  ($\Leftarrow$) Suppose there exists $S' \subseteq \{1, \ldots, n+1\}$ such that $\sum_{i \in S'} p_i = \sum_{j \notin S'} p_i$. Since $\sum_{i=1}^{n+1} p_i = 2t$, it follows that

$$\sum_{i \in S'} p_i = \sum_{j \notin S'} p_j = t$$

  If $n+1 \notin S'$ then S' is a solution to the SUBSET-SUM instance. If $n+1 \in S'$ then $S'' = \{1, \ldots, n+1\} \setminus S'$ is a solution to the subset sum instance.

Hamiltonian path

- A Hamiltonian path in a graph $G = (V, E)$ is a path that visits each vertex exactly once. The decision problem HAM-PATH is defined as

  Instance: A directed graph $G = (V, E)$, and vertices $s, t \in V$
  Question: Does there exist a Hamiltonian path from s to t in G?

- **Theorem:** HAM-PATH is **NP**-complete.
- Proof: see textbook, pages 262-267.