Recall the PRODUCT-ALLOCATION decision problem from assignment 3.

Instance: $t_1, \ldots, t_n, M, T \in \mathbb{N}$
Question: Is there an allocation $\varphi : \{1, \ldots, n\} \mapsto \{1, \ldots, M\}$ such that for every $1 \leq j \leq M$

$$\sum_{i:\varphi(i)=j} t_i \leq T$$

The corresponding optimization problem PRODUCT-ALLOCATION-OPT is, given manufacturing times $t_1, \ldots, t_n$ and a number M of factories, find an allocation of products to factories that minimizes the maximum completion time.

Instance: $t_1, \ldots, t_n, M \in \mathbb{N}$
Solution: An allocation $\varphi : \{1, \ldots, n\} \mapsto \{1, \ldots, M\}$
Objective: Minimize

$$\max_{1 \leq j \leq M} \left\{ \sum_{i:\varphi(i)=j} t_i \right\}$$

Below is an algorithm for PRODUCT-ALLOCATION-OPT, called LTF (for "Largest Times First"). The algorithm considers the products in order of non-increasing manufacturing time, and allocates each job to the factory that currently has the lowest load.

Output: $t_1, \ldots, t_n, M \in \mathbb{N}$
Input: $\varphi_1, \ldots, \varphi_n \in \{1, \ldots, M\}$ representing an allocation φ where $\varphi(i) = \varphi_i$.

sort times so that $t_1 \geq t_2 \geq \cdots \geq t_n$
$\ell_1 \leftarrow 0$, $\ell_2 \leftarrow 0$, …, $\ell_M \leftarrow 0$
for $i \leftarrow 1, \ldots, n$ **do**
  let j be such that $\ell_j = \min\{\ell_k \mid 1 \leq k \leq M\}$
  $\varphi_i \leftarrow j$
  $\ell_j \leftarrow \ell_j + t_i$
end for
return $\varphi_1, \ldots, \varphi_n$

Prove that LTF is a 2-approximation algorithm for PRODUCT-ALLOCATION-OPT.

Bonus: Prove that LTF is a $(2 - 1/M)$-approximation.