Consider the following algorithm GREEDY for KNAPSACK-OPT.

Input: $(p_1, w_1), \ldots, (p_n, w_n), W$
Output: $I \subseteq \{1, \ldots, n\}$ such that $w(I) \leq W$

sort items so that $\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \cdots \geq \frac{p_n}{w_n}$
$I \leftarrow \emptyset$
for $i \leftarrow 1, \ldots, n$ **do**
  if $w(I \cup \{i\}) \leq W$ **then**
    $I \leftarrow I \cup \{i\}$
  end if
end for

1. Prove that GREEDY does not achieve any constant approximation: that is, show that for every $\varepsilon > 0$ there is an instance of KNAPSACK-OPT such that $p(I) \leq \varepsilon \cdot p(\text{OPT})$, where I is the solution returned by GREEDY and OPT is an optimal solution.

Now consider the following modification GREEDY/MAX: run GREEDY to get a solution $I \subseteq \{1, \ldots, n\}$, and let $I' = \{j\}$, where is the index of the most profitable item that fits in the knapsack, and return the more profitable of the two solutions (i.e. either I or I').

2(a) Prove that for every $\varepsilon > 0$ there is an instance of KNAPSACK-OPT such that $p(I) \leq (1/2 + \varepsilon) \cdot p(\text{OPT})$, where I is the solution returned by GREEDY/MAX and OPT is an optimal solution.

2(b) Prove that GREEDY/MAX is a 1/2-approximation algorithm for KNAPSACK-OPT.

Hint: Consider the cases $w(I) \geq W/2$ and $w(I) < W/2$ separately, where I is the solution returned by GREEDY/MAX.