1. The instance consists of two items: the first item has profit ε and weight $\varepsilon/2$, while the second item has profit 1 and weight 1. The capacity of the knapsack is 1. GREEDY takes the small item, as it has a better ratio, and is unable to take the larger item, so it achieves a total profit of ε. The optimal solution takes the large item, for a total profit of 1. Thus if I is the solution returned by GREEDY, and OPT is the optimal solution, we have $p(I) = \varepsilon = \varepsilon \cdot p(\mathrm{OPT})$.

2(a) The instance consists of three items: the first two items each have weight $1/2$ and profit $1/2$, and the third item has weight $1/2 + \varepsilon/2$ and profit $1/2 + \varepsilon$. The capacity of the knapsack is 1. GREEDY takes the third item, and is unable to take either of the other two, so it achieves a profit of $1/2 + \varepsilon$. As the most profitable item has profit $1/2 + \varepsilon$, GREEDY/MAX also achieves a profit of $1/2 + \varepsilon$, so $p(I) = 1/2 + \varepsilon$. The optimal solution takes the first two items, so $p(\mathrm{OPT}) = 1$. Thus $p(I) = 1/2 + \varepsilon = (1/2 + \varepsilon) \cdot p(\mathrm{OPT})$.

2(b) Let I be the solution returned by GREEDY/MAX, and let OPT be an optimal solution. Let G be the solution found by GREEDY. Assume, without loss of generality, that every item is either in G or in OPT. If this is not the case, we can remove the items that are in neither set without changing anything. The idea is that, by combining MAX with GREEDY, we can assume that there is no single item whose profit is half of the optimum, or larger, because then MAX gives us a 1/2 approximation. It turns out that GREEDY does well in situations where there is no one item that is extremely profitable.

We'll use the following properties without proof.

Observation 1. *Let $S_k = \{1, 2, \ldots, k\}$. Then S_k an optimal solution for KNAPSACK with the same set of items, but capacity $w(S_k)$.*

Observation 2. *Let $S_k = \{1, 2, \ldots, k\}$. Then $p(S_k) \geq (w(S_k)/W) \cdot p(\mathrm{OPT})$.*

Case 1: OPT contains an item i with $p_i \geq p(\mathrm{OPT})/2$. In this case the MAX solution achieves a profit of at least p_i, and is therefore a 1/2 approximation. For the remaining cases we assume that there is no item whose profit is half of the optimum or larger.

Case 2: $w(G) \geq W/2$. In this case we argue that $p(G) \geq p(\mathrm{OPT})/2$. Now if GREEDY reaches capacity $W/2$ without rejecting an item, then it has achieved a profit of at least $p(\mathrm{OPT})/2$ (by observation 2). On the other hand, if GREEDY rejects an item i before reaching capacity $W/2$, then the item (which is in OPT), has weight $w_i > W/2$. Clearly there can be at most one such item, and by assumption (from case 1), we have $p_i < p(\mathrm{OPT})/2$. Let $\mathrm{OPT}' = \mathrm{OPT} \setminus \{i\}$. Then $w(\mathrm{OPT}') \leq W/2$ and $p(\mathrm{OPT}') > p(\mathrm{OPT})/2$. Now consider a modified problem in which item i is removed and the capacity is $w(G)$. By observation 1, G is optimal for this instance; and since OPT' is feasiable for this instance, we have $p(G) \geq p(\mathrm{OPT}') \geq p(\mathrm{OPT})/2$.

Case 3*:* $w(G) < W/2$. In this case, every item not in GREEDY has weight (strictly) larger than $W/2$, and thus there is at most one such item. If no such items exist, then GREEDY has taken every item, and is therefore optimal. If one such item i exists, then GREEDY has taken every item except i; and as we have assumed (from case 1) that $p_i < p(\text{OPT})/2$ it follows that the remaining items, which are in GREEDY, have profit at least $p(\text{OPT})/2$.