CSC236 / Introduction to the Theory of Computation
Midterm Test
Summer 2006

First Name: ____________________

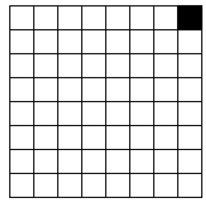
Last Name: ____________________

Student Number: ____________________

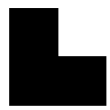
Duration: 1 hour, 50 minutes
No aids allowed

Question 1

Consider the following problem: you are given an $n \times n$ grid, where n is a power of 2, and one corner of the grid is coloured black. For example, the picture below shows such a grid for $n = 8$ (note, however, that any single corner of the grid may be coloured black, not necessarily the top-right corner).



The objective is to colour every square of the grid black, by using L-shaped tiles like the following (each such tile covers three squares of the grid). Note that tiles may not be placed in overlapping positions.



Use induction to prove that such a tiling always exists.

Hint: In the inductive step, consider how to place one tile so that the grid can be divided into four smaller grids with black corners.

Question 2

Use the Well-Ordering Principle to prove that every natural number $n \geq 2$ can be written as a *sum* of prime numbers.

Question 3

Prove partial correctness for the following program.

COUNT-ZEROES($\langle a_1, \ldots, a_n \rangle$)
Precondition: $\langle a_1, \ldots, a_n \rangle$ is a sequence of $n \geq 1$ integers
Postcondition: Returns the number of 0's in $\langle a_1, \ldots, a_n \rangle$

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z := 0
i := 1
while i <= n do
    if a_i = 0 then
        z := z + 1
    end if
    i := i + 1
end while
return z
```

Question 4

Use structural induction to prove that for every propositional formula P, there is a logically equivalent formula Q in which negation is applied only to variables (i.e. not to larger subformulas). For example, Q might contain the subformula $\neg x$ (in which a variable is negated), but not $\neg(x \vee y)$.

Question 5

Recall the language of arithmetic $\mathcal{L}_A$, with function symbols $+, \cdot$ (arity 2), s (arity 1), and 0 (arity 0), and the predicate symbol $=$ (arity 2).

(a) Express each of the following statements in $\mathcal{L}_A$. You may use the formulas $\textsc{Factor}(f, x)$, $\textsc{Prime}(x)$, and $\textsc{Less}(x, y)$ that we discussed in class.

- "There is no largest prime number"

- "Every composite number $n > 1$ can be written as the sum of two primes" (a number is composite if it is not prime)

(b) Show that the following first-order formula over $\mathcal{L}_A$ is not a tautology, by defining a structure which falsifies it.

$$\forall x \forall y (s(x) = s(y) \rightarrow x = y) \rightarrow \forall x (s(x) \neq 0)$$