CSC236 / Introduction to the Theory of Computation
Homework #1
Due date: June 27

This homework assignment consists of two parts; each part is to be handed in separately. On the course web page, you will find cover sheets for both parts. Before submitting your homework, print out both cover sheets, fill out the required information, and staple each cover sheet to the front of the corresponding part of your homework.

Part 1

1. The *Principle of Double Induction* is as follows. Suppose that S is a set with the following properties: (1) $(0, y) \in S$ for all $y \in \mathbb{N}$; and (2) for all $x, y \in \mathbb{N}$, if $(x, y) \in S$ then $(x + 1, y) \in S$. Then $\mathbb{N}^2 \subseteq S$.
   (a) Prove that the Principle of Double Induction is *equivalent* to the Principle of Induction.
   (b) Describe how you could use the Principle of Double Induction to prove that a predicate $P(x, y)$ holds for all $x, y \in \mathbb{N}$.

2. Let $a \in \mathbb{R}$ be given, such that $a \neq 1$. Use induction to prove that the following identity holds for all $n \in \mathbb{N}$.
$$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$$

3. Let $P(n)$ be the predicate "it is possible to make n cents of postage using only 3-cent and 10-cent stamps." Find a number $N \in \mathbb{N}$ such that $P(n)$ holds for all $n \geq N$. Use induction to prove that $P(n)$ indeed holds for all $n \geq N$.

4. Use the Well-Ordering Principle to prove that the following statement holds for every *even* number $n \in \mathbb{N}$, $n \geq 2$: If $a_1, \ldots, a_n \in \{0, 1\}^n$ is a sequence of n bits, and $a_1 = a_n$, then the sequence contains two adjacent bits with the same value.

Part 2

5. Prove that the following program is correct with respect to its specification.

ADD(a, b)
Precondition: $a, b \in \mathbb{N}$ are represented as decimal numbers $a = a_{n-1} \cdots a_0$, $b = b_{n-1} \cdots b_0$, where $a_i, b_i \in \{0, \ldots, 9\}$ for each $0 \leq i < n$.
Postcondition: Return $a + b$, represented as a decimal number
$c := 0$
$i := 0$
while $i < n$ **do**
  $z_i := (a_i + b_i + c) \mod 10$
  $c := \lfloor (a_i + b_i + c)/10 \rfloor$
  $i := i + 1$
end while
$z_n := c$
return $z_n \cdots z_0$

Recall that the number represented by a string of decimal digits $x_{n-1} \cdots x_0$ is $\sum_{i=0}^{n-1} x_i \cdot 10^i$.

6. Consider the following recurrence defining a function $T : \mathbb{N}^+ \mapsto \mathbb{N}$.

$$T(n) = \begin{cases} c, & \text{if } n = 1 \\ 4 \cdot T(\lceil \frac{n}{2} \rceil) + \lceil \log_2 n \rceil, & \text{otherwise} \end{cases}$$

Find an exact closed-form for $T(n)$ when n is a power of 2, and prove that your closed form is correct. Simplify your closed form as much as possible (by collecting like terms). You will need to use the following identity (for $a \neq 1$):

$$\sum_{i=1}^{n} i \cdot a^i = \frac{n \cdot a^{n+2} - (n+1) \cdot a^{n+1} + a}{(a-1)^2}$$

7. A propositional formula is called *monotone* if it only uses the connectives $\wedge$ and $\vee$. If τ_1 and τ_2 are truth assignments, then $\tau_1 \leq \tau_2$ iff $\tau_1(x) \leq \tau_2(x)$ for every variable $x \in V$.

   Use structural induction to prove that the following statement holds for every monotone formula ϕ over a set V of variables: If τ_1, τ_2 are truth assignments for ϕ such that $\tau_1 \leq \tau_2$, then $\tau_1^*(\phi) \leq \tau_2^*(\phi)$.

8. Let $p : \{0,1\}^3 \mapsto \{0,1\}$ be the *parity* function on three variables: $p(x,y,z) = 1$ if an odd number of its inputs are equal to 1, and $p(x,y,z) = 0$ otherwise.

   (a) Write down the truth table for p

   (b) Find a formula ϕ in DNF that realizes p

   (c) Find a formula ψ in CNF that realizes p