Misrepresentation in District Voting

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The problem UK

1951 UK elections:

Popular vote: 48%  48.8%
Parliament seats: 321 (51.6%)  295 (47.2%)
The problem US

1876 US elections:

Popular vote: 47.9% 50.9%

Electoral votes: 185 184
The problem US

2000 US elections:

Popular vote: 47.9% 48.4%
Electoral votes: 271 266
District voting setup

Set $C$ of $m$ candidates.

Set $V$ of voters divided into a partition $D_1, \ldots, D_z$ of equal size, so that each district has $n$ voters.
District voting setup

Each district uses voting rule $f$ to determine the winner.

The candidate that wins over the plurality of the districts is the winner of the overall election.
Score-monotone voting rules

A voting rule $f$ is score-monotone if it assigns some type of score to a candidate, and selects the candidate maximizing/minimizing this score.

E.g.: Scoring rules
Copeland
Maximin
...

Price of districting

How much are voters being misrepresented?
(for score-based voting rules $f$)

$$\max_{i \in C} \frac{\text{score of candidate } i \text{ in } f(V)}{\text{score of winning candidate in } f(V)}$$
Plurality 2 candidates

2 $\ell+1$ districts, each with $2t+1$ voters

$\ell+1$ districts

\[ \begin{align*}
\text{voters} & : t+1 & \quad a & \quad a & \quad a & \quad a \\
\quad b & \quad b & \quad b & \quad b \\
\end{align*} \]

$\ell$ districts

\[ \begin{align*}
\text{voters} & : t & \quad a \\
\quad b & \quad b & \quad b & \quad b & \quad b \\
\quad b \\
\end{align*} \]
Plurality $m$ candidates
Plurality $m$ candidates

$1 + \frac{n - 2\left\lfloor \frac{n}{2} \right\rfloor + 1}{q + 2} + \frac{(z + 1)(\left\lceil \frac{n}{2} \right\rceil - 1) - n}{(\ell + 2)(q + 2)} \approx \Theta(m^2)$

$q = \left\lfloor \frac{n}{m} \right\rfloor$
Plurality majority twist

$m$ candidates

\[
\frac{q + 1}{q + 2} + \frac{n(\left\lceil \frac{n}{2} \right\rceil - 1)}{(q + 2)(\left\lceil \frac{n}{2} \right\rceil + 1)} \approx \Theta(m)
\]

\[q = \left\lfloor \frac{n}{m} \right\rfloor\]
Other scoring rules

\( k \)-approval: \( \Theta(m^2/k) \)

Veto: \( \Theta(m) \)

Borda: \( \Theta(m^2) \)
Copeland

21 voters

20 voters

District winner is \( a \).

Copeland winner is \( b \) with score 2,
\( a \) with score 0.
Copeland Price of districting

\[
\max_{i \in C} \frac{\text{score of candidate } i \text{ in } f(V) + m}{\text{score of winning candidate in } f(V) + m}
\]

District winner may have worst possible score, while Copeland winner has best possible score.
Simulations: Borda uniform

Voters in each district
Simulations: plurality Mallows

The diagram shows the results of simulations for plurality Mallows under different conditions. The x-axis represents the number of voters in each district, ranging from 0 to 5000. The y-axis represents the voting percentage, ranging from 1.02 to 1.14.

Different symbols are used to represent different districts:
- Blue dots for district 3
- Red dots for district 4
- Green dots for district 5
- Violet dots for district 6
- Cyan dots for district 7

The data points indicate the voting percentages for each district under varying voter counts.
Simulations: Copeland Mallows
What’s next?

Another paper with Yoad…
(complexity, geography, real world data)

More voting methods

Is Homogeneity/heterogeneity of districts good or bad?

More effects of districts on outcomes and their representability.
Fin

Thanks for listening!