Cooperative Weakest Link Games

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Weakest link scenarios

Cooperative games

I is a group consisting of n agents.

$v{:}2^I \longrightarrow \mathbb{Q}$ gives each coalition of subsets of I a value.

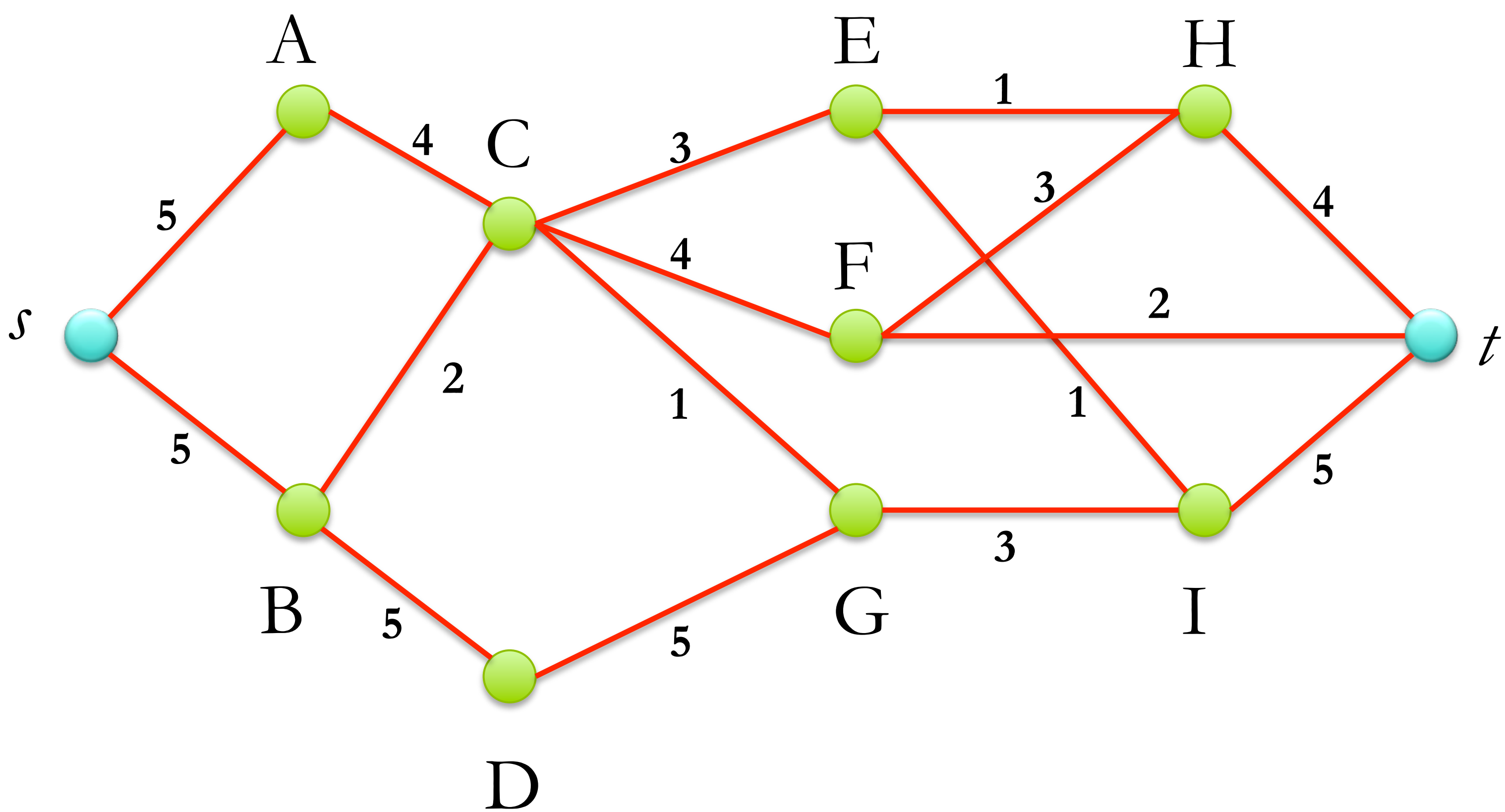
An **imputation** $(p_1,\ldots,p_n)$ divides the value of the grand coalition (i.e., $v(I)$) between the varioius agents.

Weakest link games

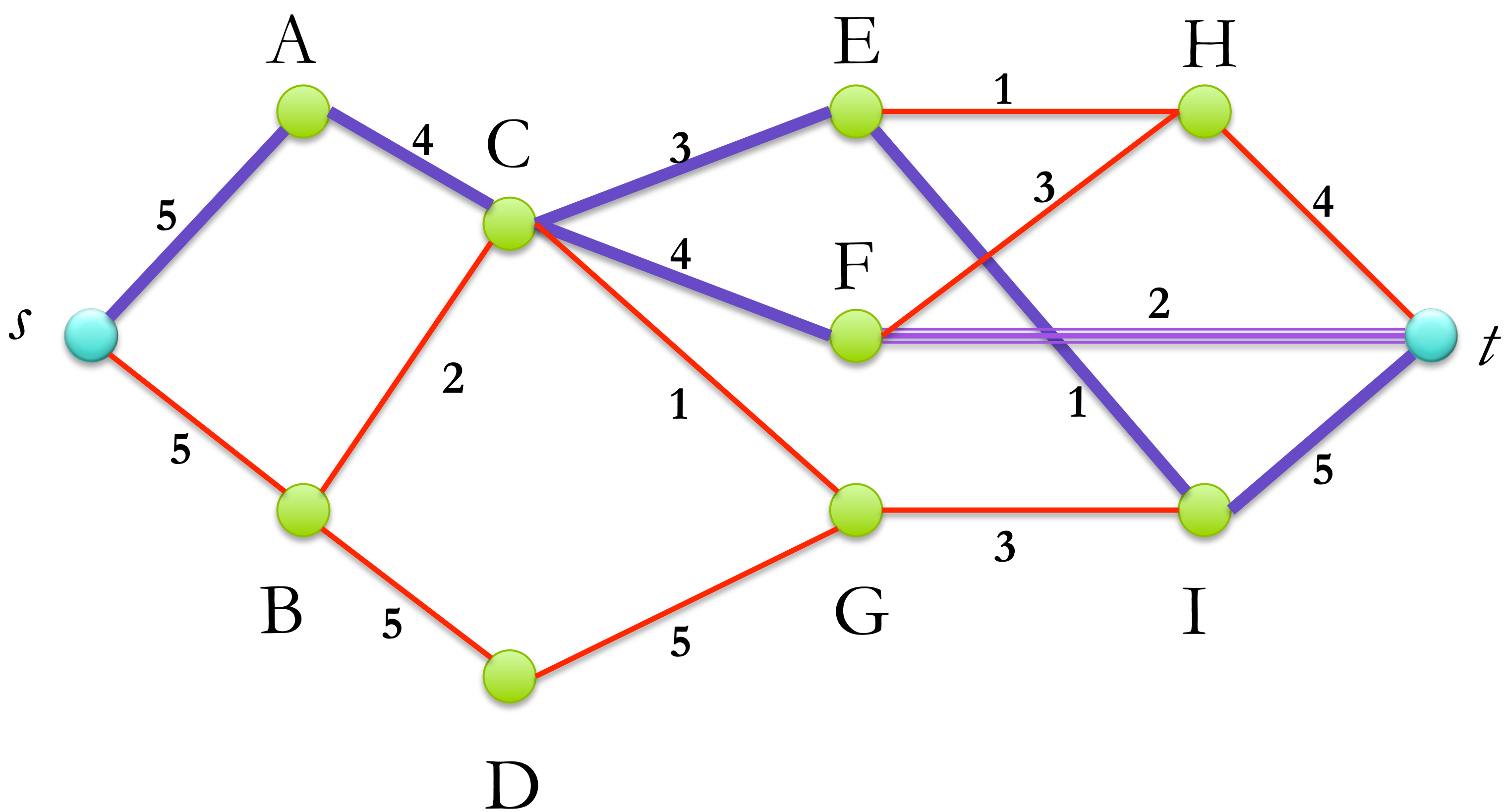
A graph $G=(V,E)$ with weighted edges and with two special vertices – s and t.

Value of a coalition is calculated by taking all paths between s and t which is included in the coalition. The value of each path is the weight of the minimal edge. The coalitions's value is the maximal of the paths.

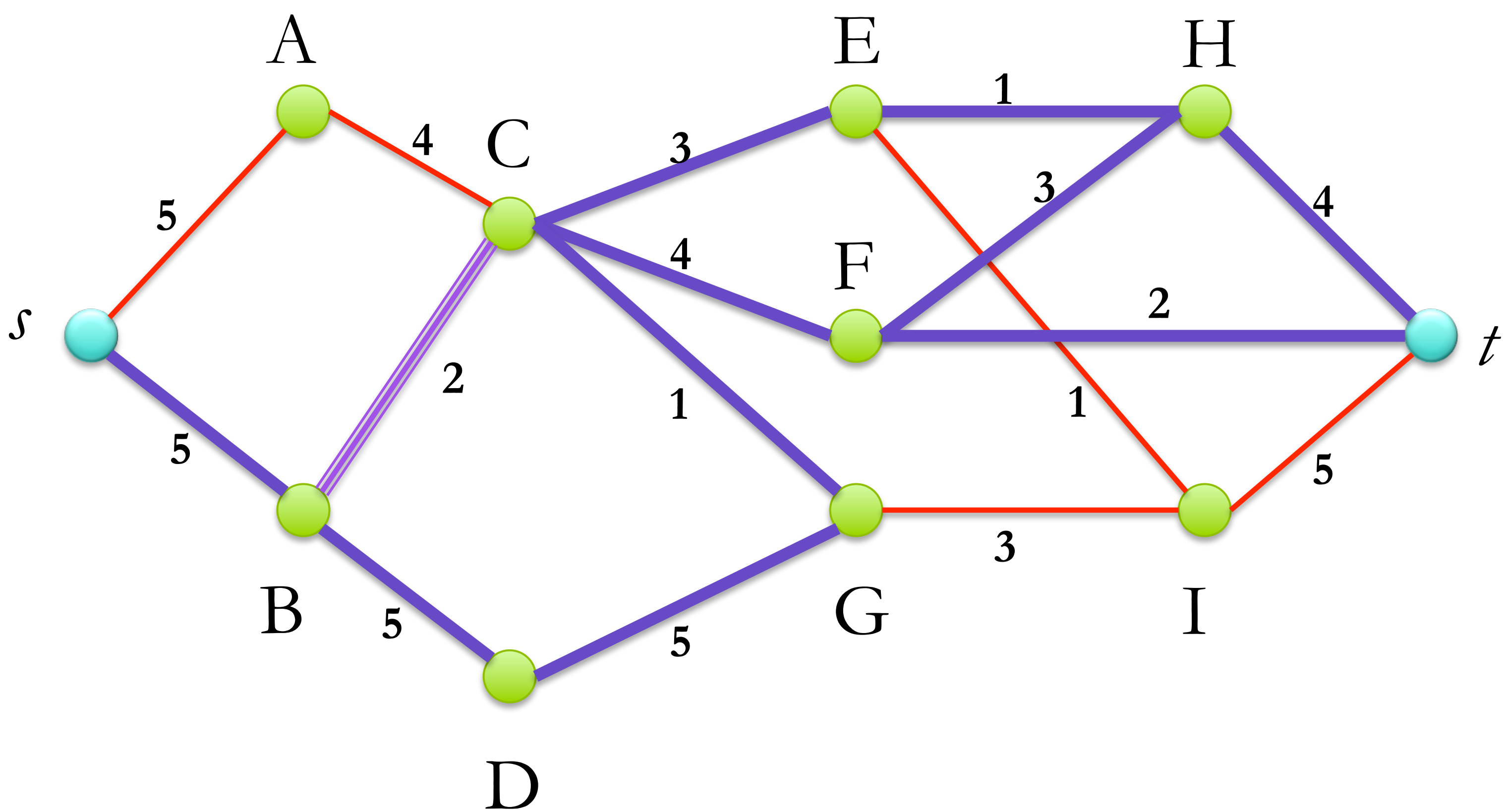
Weakest link games example



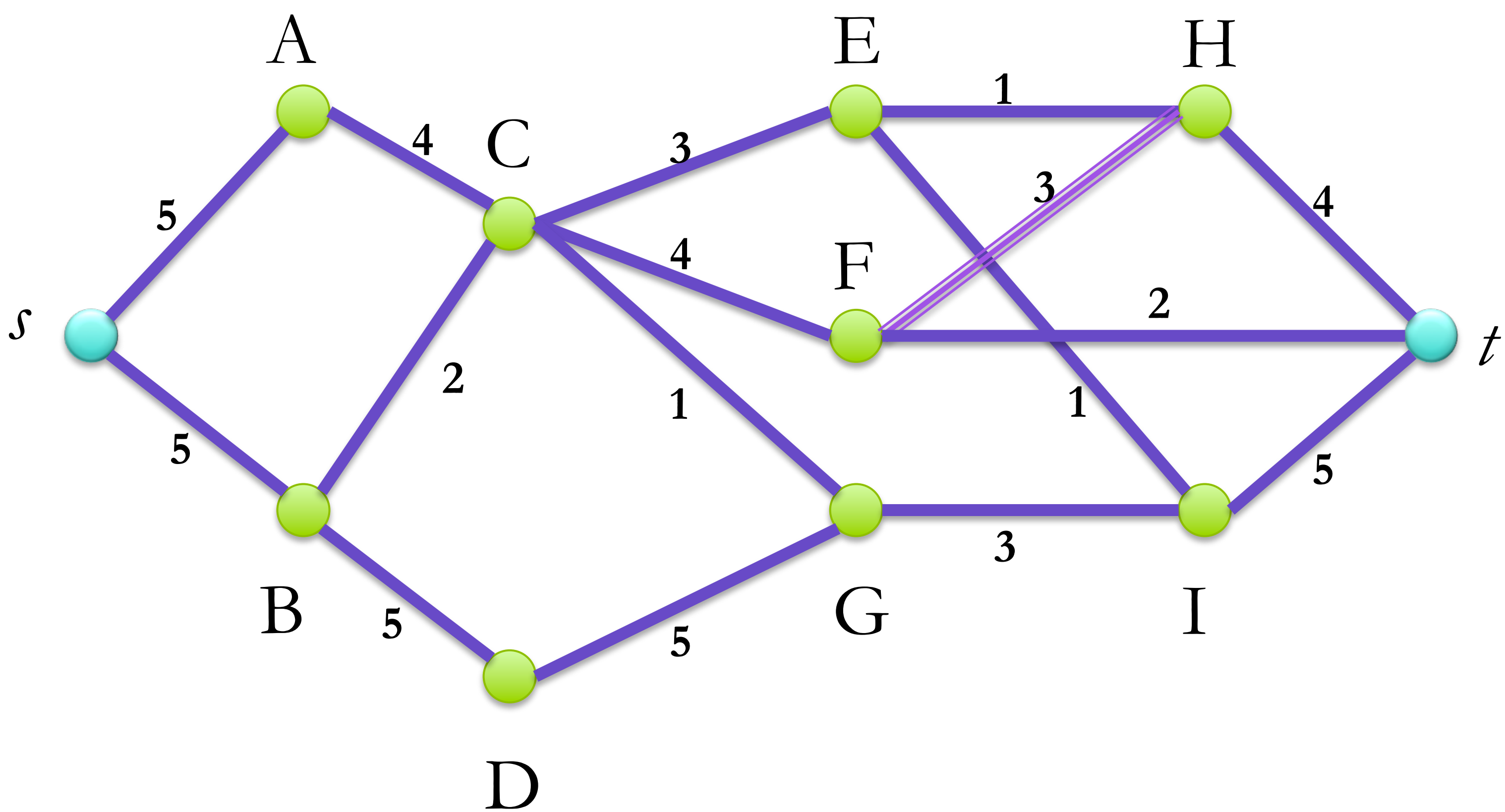
Weakest link games example



Weakest link games example



Weakest link games example



Core and ε-core definitions

Core is the set of imputations that aren't blocked by any coaliton (i.e., no subset of agents has an incentive to leave)

ε-core, for $\varepsilon>0$, is the set of imputations that, for each subset of agents C, gives its members at least $v(C)-\varepsilon$.

Core – theorem I

Calculating the value of a coalition C is polynomial

For every value of edge weight τ we build a graph with edges of minimal weight τ, and see (using DFS) if a path still exists between s and t.

Core – theorem II

Testing if an imputation is in the core (or ε-core) is polynomial

For every value of edge weight τ we build a graph with edges of minimal weight τ, and modify the weight of each edge to be its imputation. We find the shortest path, and if its total weight is below τ, we have a blocking coalition.

Core – theorem III

Emptiness of core (or ε-core) is polynomial

Using previous slide's algorithm as a separation oracle, we utilize the Ellipsoid method to solve the linear program $\forall C \subset I : \sum_{i \in C} p_i \geq v(C)$

$$\sum_{i \in I} p_i = v(I)$$

Optimal coalition structure

definition

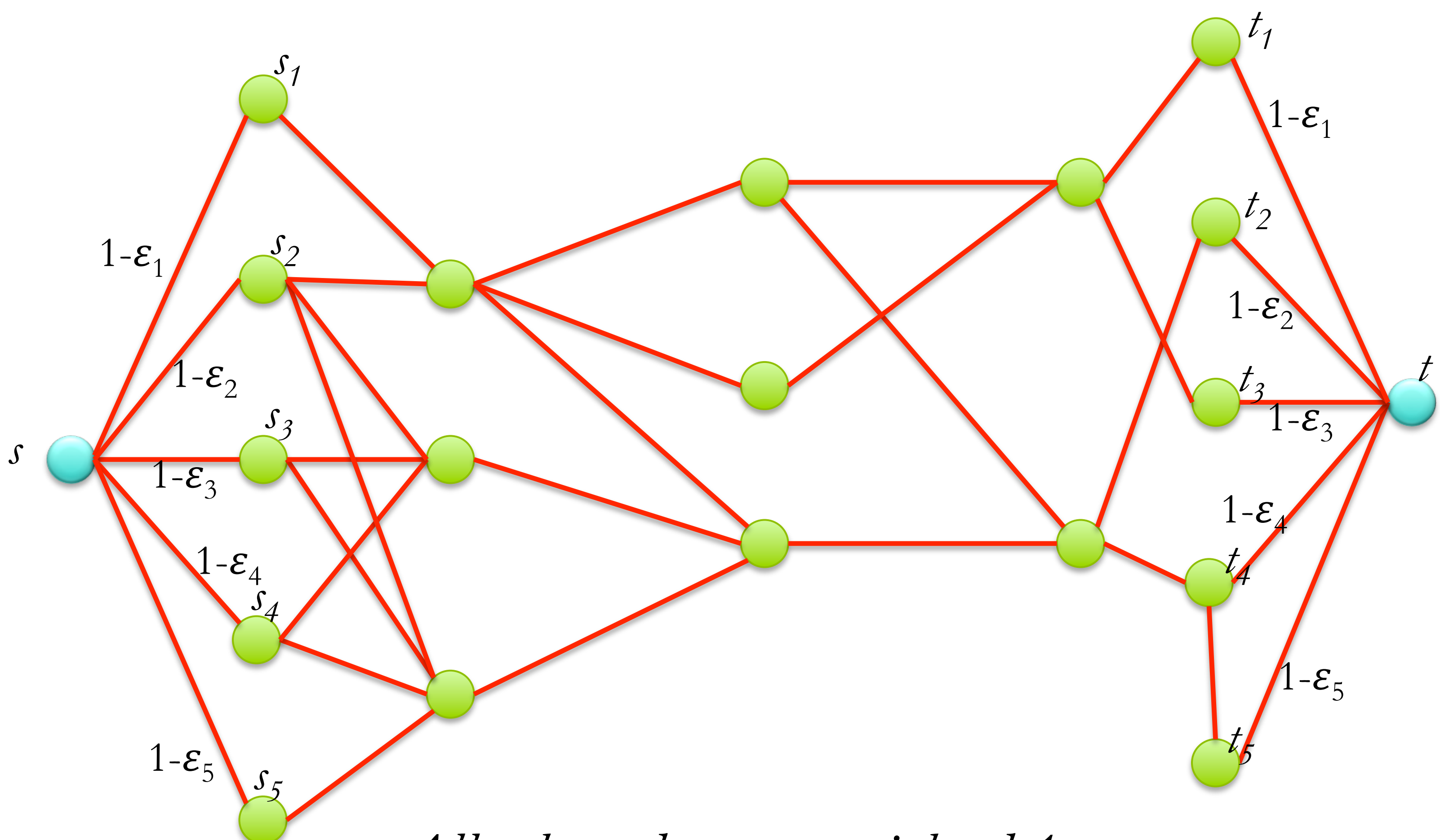
A **coalition structure** is a partition of the agents into disjoint groups, with the value of the structure being the sum of the values of each group.
The **optimal coalition structure** is the the partition with the maximal value.

Optimal coalition structure – theorem I

Finding if the value of a the optimal coalition structure is above k is NP-hard

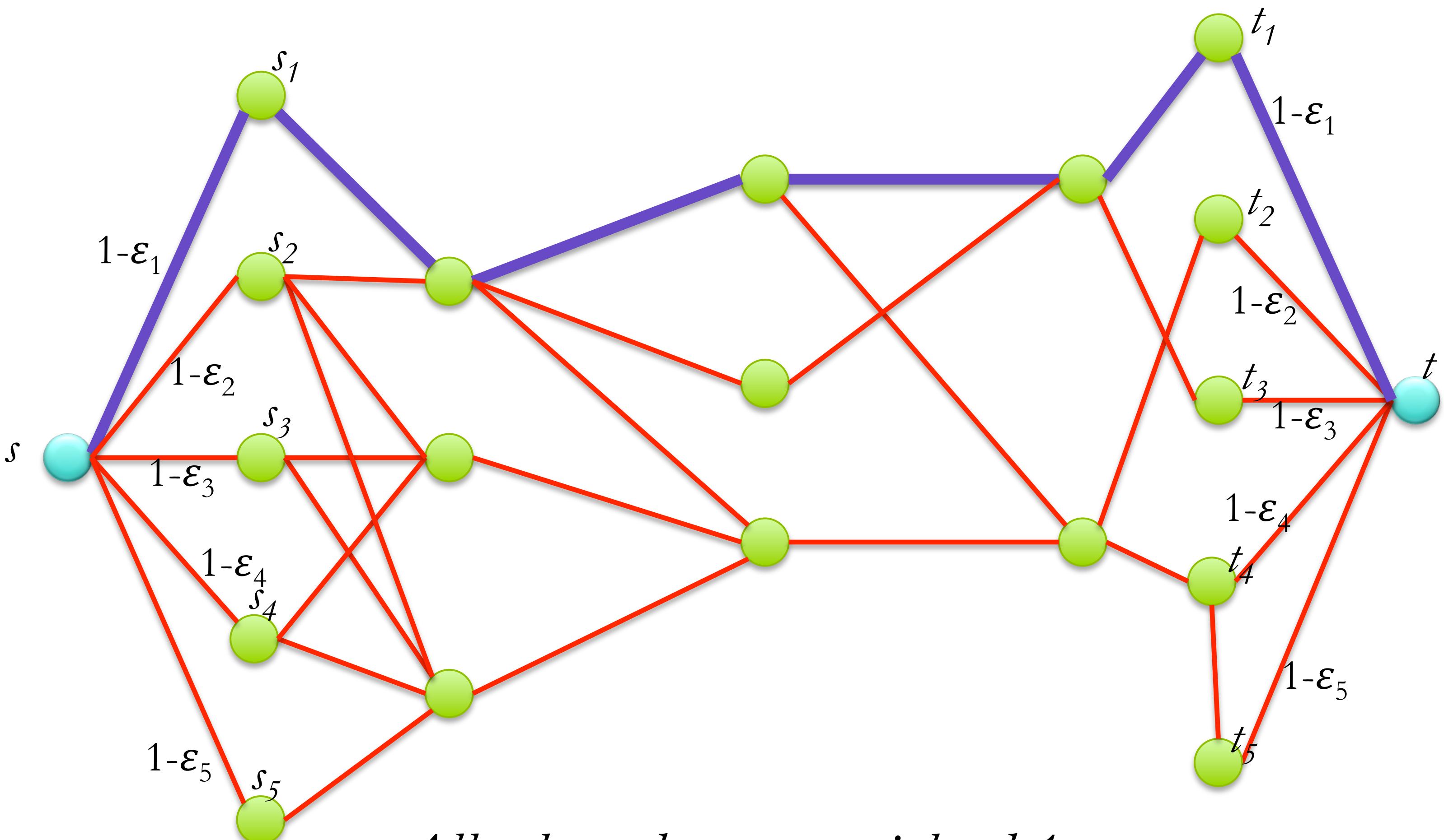
A reduction from Disjoint Paths Problem: set of k pairs of (s_i,t_i), where we wish to know of there are k disjoint paths such that the i'th path connects s_i to t_i.

Optimal coalition structure – theorem I reduction



All other edges are weighted 1

Optimal coalition structure – theorem I reduction



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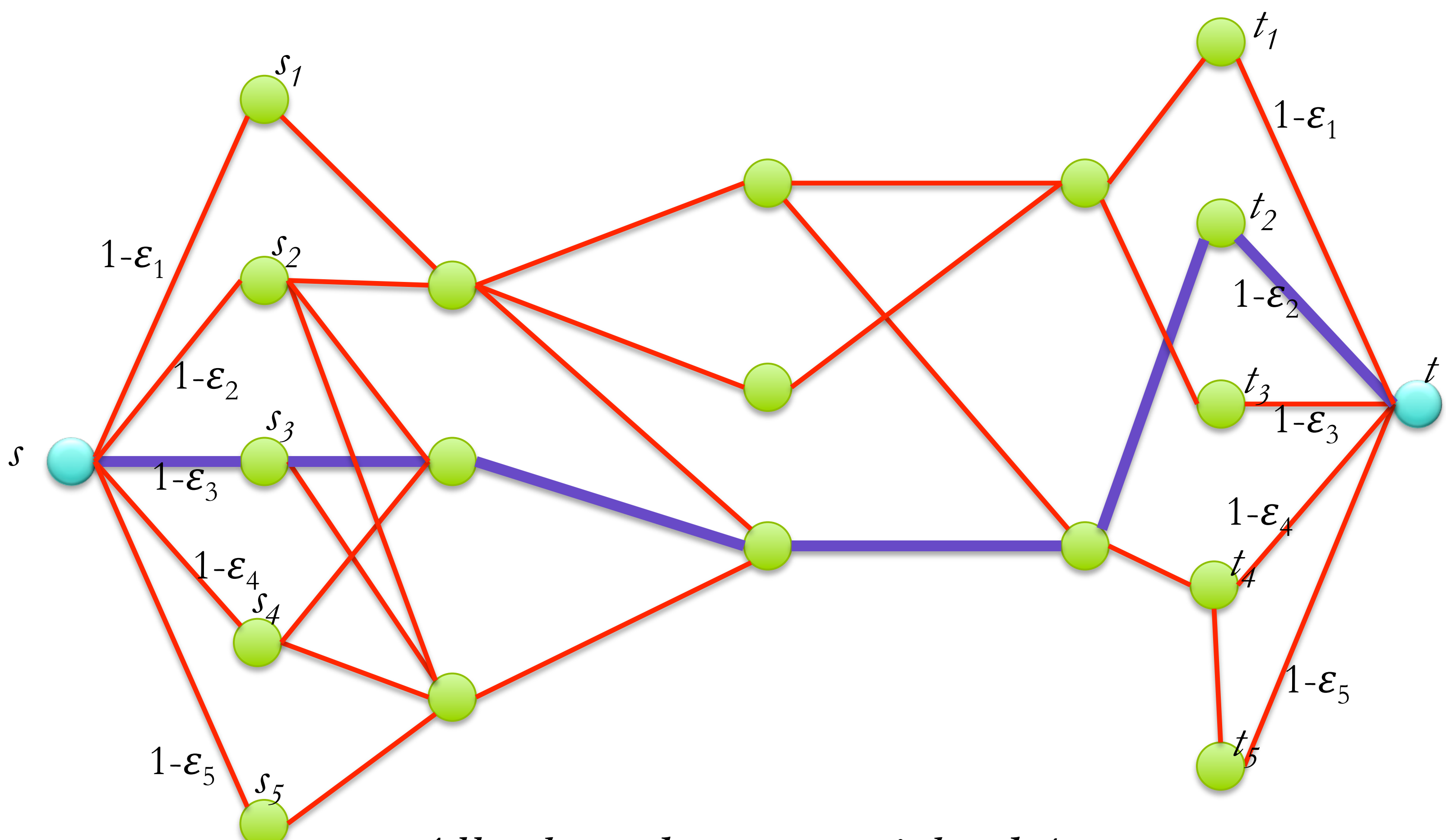
Optimal coalition structure – theorem I reduction



All other edges are weighted 1

Optimal coalition structure – theorem II

There is an O(log n) approximation to the optimal coalition structure problem

For each τ, we can find (using max-flow algorithms) the number of disjoint paths with value of at least τ - n_τ. We maximize τn_τ, and that can be shown to be an O(log n) of the result…

Optimal coalition structure – theorem II

Let *w'* be *v(I)*.

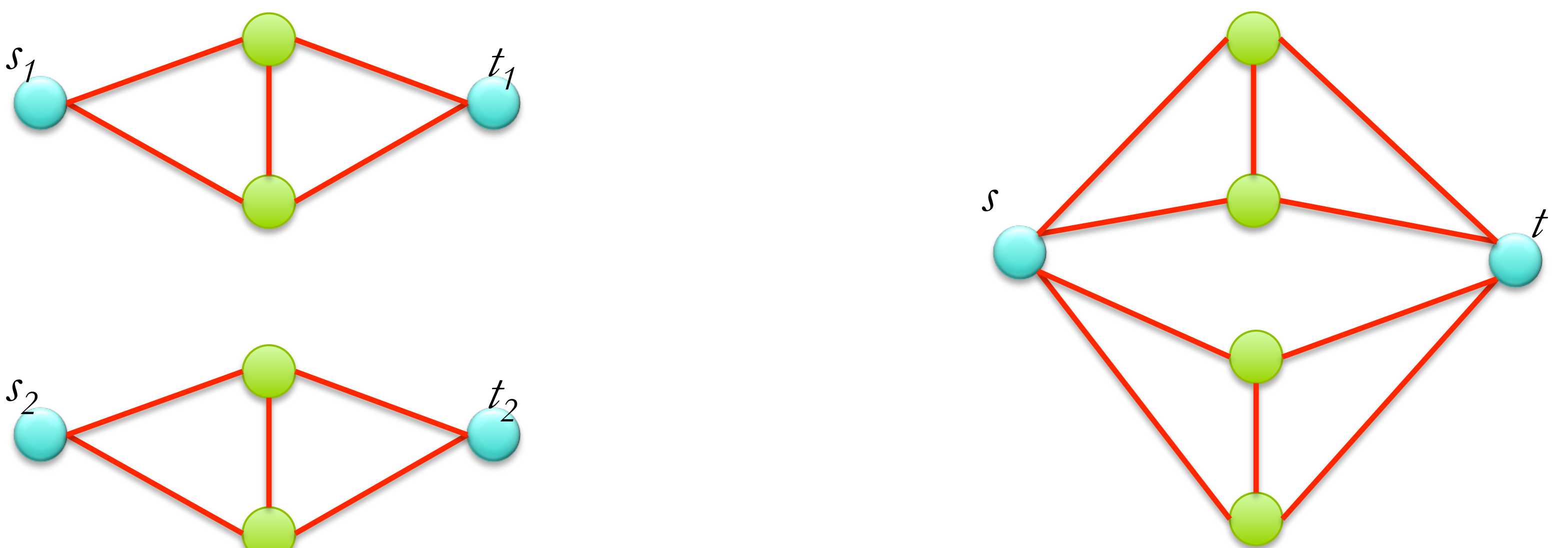
Hence, the optimal coalition structure is less than $\sum_{i=1}^{\infty} n_i \frac{w'}{2^{i-1}}$. It's easy to see that $\sum_{i=2\log(n)}^{\infty} n_i \frac{w'}{2^{i-1}}$ is less than $2\frac{w'}{n}$, and hence, somewhere from 1<*i*<2*log*(n), there is an *i* which is worth $\frac{1-\frac{2}{n}}{2\log(n)}$.

Cost of Stability definition

Cost of Stability is the minimal amount needed to be added to $v(I)$ in order to make some imputation of the grand coalition be in the core.

Cost of stability joining graphs

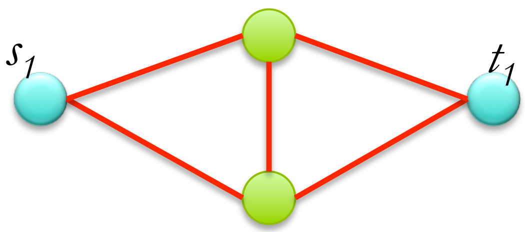
Parallel composition:

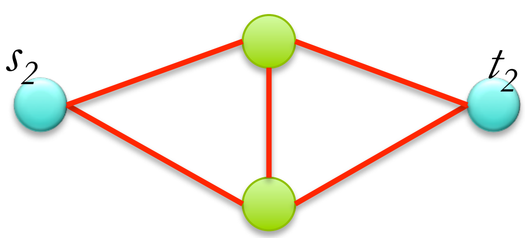


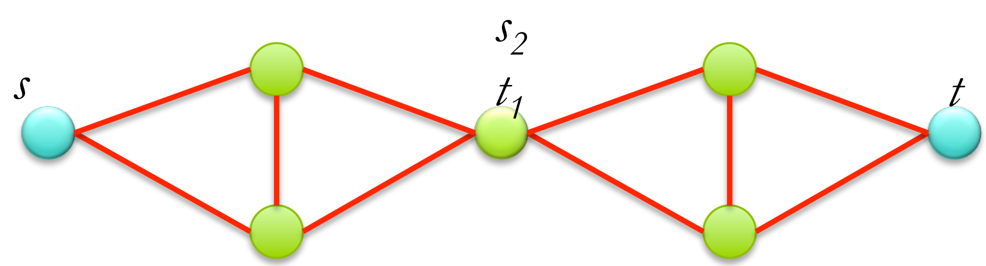
$$\sum_{G_i} (CoS(G_i) + v_i(G_i)) - \max_{G_i}(v_i(G_i))$$

Cost of stability joining graphs

Serial composition:







$$\min_i CoS(G_i^{\min_{j\neq i}(v(G_j))})$$

Where G_i^j is G_i where all edges above j lowered to j.

Future directions

More solution concepts (nucleolus)

Power indices

A restricted class where optimal coalition structure can be solved

Uncertainty in agent behaviour

Weakest link without graphs?

I am the weakest link, goodbye.

Thanks for listening!