Cooperative Weakest Link Games

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Cooperative games

$I$ is a group consisting of $n$ agents.

$v:2^I \rightarrow \mathbb{Q}$ gives each coalition of subsets of $I$ a value.

An **imputation** $(p_1, \ldots, p_n)$ divides the value of the grand coalition (i.e., $v(I)$) between the various agents.
A graph $G=(V,E)$ with weighted edges and with two special vertices – $s$ and $t$.

Value of a coalition is calculated by taking all paths between $s$ and $t$ which is included in the coalition. The value of each path is the weight of the minimal edge. The coalition’s value is the maximal of the paths.
Weakest link games example
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A 5 5 2 4 5 5 3
B 5 5 4 3 1 3 4
C 4 3 1 2 1 2 5
D 5 5 4 3 1 3 5
E 1 1 3 3 2 2 5
F 5 3 4 4 5 2 5
G 5 5 1 5 3 3 5
H 1 1 5 5 5 5 5
I 1 2 5 5 5 5 5

s t
Core and $\varepsilon$-core definitions

Core is the set of imputations that aren’t blocked by any coalition (i.e., no subset of agents has an incentive to leave).

$\varepsilon$-core, for $\varepsilon > 0$, is the set of imputations that, for each subset of agents $C$, gives its members at least $\nu(C) - \varepsilon$. 
Calculating the value of a coalition C is polynomial

For every value of edge weight $\tau$ we build a graph with edges of minimal weight $\tau$, and see (using DFS) if a path still exists between $s$ and $t$. 
Core – theorem II

Testing if an imputation is in the core (or $\varepsilon$-core) is polynomial

For every value of edge weight $\tau$ we build a graph with edges of minimal weight $\tau$, and modify the weight of each edge to be its imputation. We find the shortest path, and if its total weight is below $\tau$, we have a blocking coalition.
Emptiness of core (or $\varepsilon$-core) is polynomial

Using previous slide’s algorithm as a separation oracle, we utilize the Ellipsoid method to solve the linear program

$$\forall C \subset I : \sum_{i \in C} p_i \geq v(C)$$

$$\sum_{i \in I} p_i = v(I)$$
A coalition structure is a partition of the agents into disjoint groups, with the value of the structure being the sum of the values of each group. The optimal coalition structure is the partition with the maximal value.
Finding if the value of a the optimal coalition structure is above $k$ is NP-hard

A reduction from Disjoint Paths Problem: set of $k$ pairs of $(s_i, t_i)$, where we wish to know if there are $k$ disjoint paths such that the $i$th path connects $s_i$ to $t_i$. 
Optimal coalition structure – theorem I

- All other edges are weighted 1
Optimal coalition structure – theorem I

- Reduction

All other edges are weighted 1
Optimal coalition structure – theorem 1

\[ s_1, s_2, s_3, s_4, s_5 \rightarrow t \]

All other edges are weighted 1
There is an $O(\log n)$ approximation to the optimal coalition structure problem.

For each $\tau$, we can find (using max-flow algorithms) the number of disjoint paths with value of at least $\tau - n_\tau$. We maximize $\tau n_\tau$, and that can be shown to be an $O(\log n)$ of the result…
Let $w'$ be $v(I)$.

Hence, the optimal coalition structure is less than $\sum_{i=1}^{\infty} n_i \frac{w'}{2^{i-1}}$. It’s easy to see that $\sum_{i=2 \log(n)}^{\infty} n_i \frac{w'}{2^{i-1}}$ is less than $2 \frac{w'}{n}$, and hence, somewhere from $1 < i < 2 \log(n)$, there is an $i$ which is worth $\frac{1 - \frac{2}{n}}{2 \log(n)}$. 
Cost of Stability is the minimal amount needed to be added to $\nu(I)$ in order to make some imputation of the grand coalition be in the core.
Cost of stability joining graphs

Parallel composition:

\[ \sum_{G_i} (\text{CoS}(G_i) + \nu_i(G_i)) - \max_{G_i} \nu_i(G_i) \]
Cost of stability joining graphs

Serial composition:

\[
\min_i \text{COS}(G_{i, \min_{j \neq i}}(v(G_j)))
\]

Where \(G_{i,j}\) is \(G_i\) where all edges above \(j\) lowered to \(j\).
Future directions

More solution concepts (nucleolus)

Power indices

A restricted class where optimal coalition structure can be solved

Uncertainty in agent behaviour

Weakest link without graphs?
I am the weakest link, goodbye.

Thanks for listening!