1. THE O STATEMENT

Let $\mathcal{F}$ be the set of functions from $\mathbb{N} \to \mathbb{R}^+$. For $f, g \in \mathcal{F}$, let $g \in O(f)$ be defined by:

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n).$$

(1) Write out the proof structures for $g \in O(f)$, and the negation $g \notin O(f)$.
(2) Let $f, g \in \mathcal{F}$ be defined by $f(n) = n^2, g(n) = 165n^2$ for all $n \in \mathbb{N}$. Prove that $g \in O(f)$.
(3) Let $f, g \in \mathcal{F}$ be defined by $f(n) = n^3, g(n) = \frac{n^3}{165}$ for all $n \in \mathbb{N}$. Prove that $g \in O(f)$.
(4) Let $f, g \in \mathcal{F}$ be defined by $f(n) = n^4, g(n) = n^3 + 165$ for all $n \in \mathbb{N}$. Prove that $g \in O(f)$.
(5) Let $f, g \in \mathcal{F}$ be defined by $f(n) = n^2, g(n) = n^2 + 2n - 165$ for all $n \in \mathbb{N}$. Prove that $g \in O(f)$.
(6) Let $f, g \in \mathcal{F}$ be defined by $f(n) = \begin{cases} 0, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$, $g(n) = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$. Prove that $g \notin O(f)$. 


1. The O Statement

The exercises are fairly standard. Remind them that generating proof structures is mechanical, and they should practice until they can do it quickly and easily.

Here’s the heart of the actual proofs (but the students should put them in the proof structure):

(2) \( c = 165, \ B = 0. \)

\[ g(n) = 165n^2 = cn^2 = cf(n) \leq cf(n) \]

The proof goes as follows:

Let \( c = 165. \) Then \( c \in \mathbb{R}^+. \)

Let \( B = 0. \) Then \( B \in \mathbb{N}. \)

Let \( n \in \mathbb{N} \) be arbitrary.

Assume \( n \geq B. \)

Then \( g(n) = 165n^2 = cn^2 = cf(n) \leq cf(n). \)

Thus \( g(n) \leq cf(n). \)

Thus \( n \geq B \Rightarrow g(n) \leq cf(n). \)

Since \( n \) is an arbitrary element of \( \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n). \)

So \( \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n). \)

So \( \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n). \)

(3) \( c = 1, \ B = 0. \)

\[ n^3 / 165 \leq n^3 = 1 \cdot n^3 \]

I know they’ll all take \( c = 1/165, \) but it’s worth showing them this.

(4) \( c = 166, \ B = 1. \)

\[ n^3 + 165 \leq n^3 + 165n^3 \leq 166n^3 \leq 166n^4 \]

(5) \( c = 3, \ B = 1. \)

\[ n^2 + 2n - 165 \leq n^2 + 2n \leq n^2 + 2n^2 \leq 2n^2 \text{ (since } n \geq b = 1) \]

\[ \ldots \]

(6) No matter what \( c \) or \( B \) is chosen, there is always an even \( n \geq B \) such that \( g(n) = 1 \nless cf(n) = 0. \)