1. Course Prerequisites

Let \( C \) be a set of courses, and let \( P(x, y) \) mean course \( x \) is a prerequisite for course \( y \). Rewrite the following symbolically:

1. “There is no prerequisite for CSC108.”
2. “Every course has a prerequisite.”
3. “Some course is not a prerequisite for any course.”
4. “No course is a prerequisite for itself.”
5. “Some courses have several prerequisites.”
6. “No course has more than two prerequisites.”
7. “Some courses have the same prerequisites.”

2. Distributive Laws for Quantifiers?

Which are true, which are true in one direction, and which are false both directions? Explain your answers.

1. \( \forall x \in D, P(x) \land Q(x) \iff (\forall x \in D, P(x)) \land (\forall x \in D, Q(x)) \)
2. \( \exists x \in D, P(x) \land Q(x) \iff (\exists x \in D, P(x)) \land (\exists x \in D, Q(x)) \)
3. \( \forall x \in D, P(x) \lor Q(x) \iff (\forall x \in D, P(x)) \lor (\forall x \in D, Q(x)) \)
4. \( \exists x \in D, P(x) \lor Q(x) \iff (\exists x \in D, P(x)) \lor (\exists x \in D, Q(x)) \)
(1) “There is no prerequisite for CSC108.”
Sample solution: \( \forall x \in C, \neg P(x, \text{CSC108}) \)

(2) “Every course has a prerequisite.”
Sample solution: \( \forall x \in C, \exists y \in C, P(y, x) \)

(3) “Some course is not a prerequisite for any course.”
Sample solution: \( \exists x \in C, \forall y \in C, \neg P(x, y) \)

(4) “No course is a prerequisite for itself.”
Sample solution: \( \forall x \in C, \neg P(x, x) \)

(5) “Some courses have several prerequisites.”
Sample solution: \( \exists x \in C, \exists y \in C, \exists z \in C, P(y, x) \land P(z, x) \land y \neq z \)

(6) “No course has more than two prerequisites.”
Sample solution:
\[ \forall x \in C, \forall y \in C, \forall z \in C, \forall w \in C, (P(x, w) \land P(y, w) \land P(z, w)) \Rightarrow (x = y \lor x = z \lor y = z) \]

(7) “Some courses have the same prerequisites.”
Sample solution: \( \exists x \in C, \exists y \in C, \forall z \in C, P(z, x) \iff P(z, y) \)

Which are true, which are true in one direction, and which are false both directions? Explain your answers.

(1) \( \forall x \in D, P(x) \land Q(x) \iff (\forall x \in D, P(x)) \land (\forall x \in D, Q(x)) \)
Sample solution: True. Thought of as sets, the left-hand side says that all of \( D \) is in the intersection of \( P \) and \( Q \), which is the same as saying that all of \( D \) is in \( P \) and all of \( D \) is in \( Q \).

(2) \( \exists x \in D, P(x) \land Q(x) \iff (\exists x \in D, P(x)) \land (\exists x \in D, Q(x)) \)
Sample solution: The left-hand claim implies the right-hand claim. If there is an element of the domain for which \( P \) and \( Q \) are jointly true, then that same element provides an example where \( P \) is true, and the same element provides an example where \( Q \) is true. The right-hand claim doesn’t imply the left-hand. As a counter-example, consider \( D = \mathbb{N} \), \( P(n) \): “\( n \) is odd”, and \( Q(n) \): “\( n \) is even.” In this case the right-hand claim is true: I can find an even natural number, and I can find an odd natural number. However, the left-hand claim is false: I can’t find a natural number that is simultaneously even and odd.

(3) \( \forall x \in D, P(x) \lor Q(x) \iff (\forall x \in D, P(x)) \lor (\forall x \in D, Q(x)) \)
Sample solution: The right-hand claim implies the left-hand claim. If the right-hand claim is true, there are two cases to consider. In the first case, \( P \) is true of every element of the domain, so it follows that \( P \lor Q \) is true of every element of the domain. In the second case, \( Q \) is true of every element of the domain, so it follows that \( P \lor Q \) is true of every element of the domain. However, the left-hand claim doesn’t imply the right-hand claim. As a counter-example, consider (again) \( D = \mathbb{N} \), \( P(n) \): “\( n \) is odd”, and \( Q(n) \): “\( n \) is even.” Now the left-hand claim is true, whereas the right-hand claim is false.

(4) \( \exists x \in D, P(x) \lor Q(x) \iff (\exists x \in D, P(x)) \lor (\exists x \in D, Q(x)) \)
Sample solution: This is true. Thought of as sets, the left-hand side says that the union of \( P \) and \( Q \) is non-empty, which is true iff \( P \) is non-empty or \( Q \) is non-empty (which is what the right-hand side says).