CSC 165
quantifiers

reminder

- why CSC165? ambiguity, precision, etc.
- universal quantification, $\forall -$ negation?
- existential quantification, $\exists -$ negation?
- set properties and verification
- symbols e.g. $L(x) : x \in L$
- sentences and statements (does not refer to unquantified variables)
  example:
  - The employee earns over 55,000.
  - Every employee makes less than 55,000.
quantifiers as sets

Think of quantification as set operations. Suppose $E$ is the set of employees, $M$ is the set of male employees, $F$ is the set of female employees, and $O$ is the set of employees earning over 42,000. Express the following in terms of set operations (subsets, complements, etc.):

- All employees earn over 42,000
- Some female employee earns over 42,000
- Some male employee earns over 42,000
- All male employees earn over 42,000
sentences

We'll use the term **sentence** to refer to expressions that are structured to evaluate to either true or false. Sometimes key objects in a sentence have not been specified, so the sentence is open, and we may not be able to evaluate it:

- The employee earns over 55,000.
- Every employee makes less than 55,000.

Quantifying an unspecified variable may change an open sentence (about some unspecified element) to a **statement** — an expression that can be evaluated to true or false.
Using symbols such as $M$ to stand for the set of male employees, and $O$ to stand for employees earning over 42,000 allows us to abstract away details and focus on the set relationship, whether $M \subseteq O$ or not.

We extend the symbolism in order to emphasize the connection between the set $L$ (employees earning less than 55,000) and the boolean function that indicates whether something is in $L$:

$$\text{def : } L(x) : x \in L$$

Notice how similar this is to the definition of a boolean function (the keyword `def` would make it even more so). The argument $x$ shows us how the argument is used in the definition. We can’t evaluate $L(x)$ until we know what $x$ is bound to.
for all, $\forall$

Change open sentence $L(x)$ into a statement by universally quantifying it. This operation is used often enough that there is a symbol provided for convenience:

$\forall$ employees, the employee makes less than 55,000.

$\forall$ employees $x$, $x$ makes less than 55,000.

$\forall x \in E, L(x)$. 
there exists, $\exists$

The corresponding existential statement about employees earning less than 55,000:

$$\exists x \in E, L(x)$$

...is not a statement about an element $x$, but about the set $E \cap L$. 
implication

There’s a couple of ways to expression the IMPLICATION

IF an employee is male, THEN he earns less than 55,000.

This could accurately be expressed using universal quantification by restricting the set we are considering:

$$\forall x \in E \cap M, L(x)$$

It’s sometimes convenient to separate the “male implies less than 55,000” from the domain “employee” — perhaps seeing how the rule holds up in the set $H$ of humans, or the set $S$ of short employees. The form “if $P$, then $Q$” is called IMPLICATION.
verifying implication

Which of the following are a counter-example to “if the employee is male, then he earns less than 55,000”?

- Carlos?
- Ellen?
- Al?
- Gwen?

<table>
<thead>
<tr>
<th>Employee</th>
<th>Gender</th>
<th>Salary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>male</td>
<td>60,000</td>
</tr>
<tr>
<td>Betty</td>
<td>female</td>
<td>500</td>
</tr>
<tr>
<td>Carlos</td>
<td>male</td>
<td>40,000</td>
</tr>
<tr>
<td>Doug</td>
<td>male</td>
<td>30,000</td>
</tr>
<tr>
<td>Ellen</td>
<td>female</td>
<td>50,000</td>
</tr>
<tr>
<td>Flo</td>
<td>female</td>
<td>20,000</td>
</tr>
<tr>
<td>Gwen</td>
<td>female</td>
<td>95,000</td>
</tr>
</tbody>
</table>
In implication “If P, then Q” we call P the **antecedent** and Q the **consequent**. Sometimes, in natural language an implication goes both ways:

If you eat your vegetables, then you can have dessert.

...but in logic, we allow the case where you don’t eat your vegetables and still are allowed to eat dessert to be consistent with the implication

Even true implication doesn’t give you causality:

If it rains today, the sun will rise tomorrow.
Here's a universally-quantified implication, where $E$ is the set of employees, $F$ the set of female employees, and $L$ the set of employees earning less than 55,000:

$$\forall x \in E, \text{ if } F(x), \text{ then } L(x).$$

If the implication is true, what can you deduce about the following sets:

1. $F$, the set of female employees?
2. $L$, the set of employees earning less than 55,000?
3. $\overline{F}$, the set of non-female employees?
4. $\overline{L}$, then set of employees earning at least 55,000?

If you could add a new employee, what gender and salary combination would you pick in order to falsify the implication?
Implication is used frequently enough to deserve its own symbol. The universally-quantified implication from the previous slide could be written:

$$\forall x \in E, F(x) \Rightarrow L(x)$$

Reverse the direction, and you have the converse of the original implication.

$$\forall x \in E, L(x) \Rightarrow F(x)$$

What connection is there between the truth of an implication and the truth of its converse? Explain.
contrapositive

contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$ Another symbol, $\neg$, toggles the truth value of a statement. When we toggle AND reverse an implication, we get its CONTRAPOSITIVE. Compare the meanings of:

\[
\forall x \in E, F(x) \Rightarrow L(x) \\
\forall x \in E, \neg L(x) \Rightarrow \neg F(x)
\]

What information does each form give you in each of the four following cases:

1. When $x \in F$?
2. When $x \not\in F$?
3. When $x \in L$?
4. When $x \not\in L$?

What would a counter-example to each form be?
The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$.
Other words: “$P$ implies $Q$” is “$Q$ implies $P$.”
An implication and its converse don’t mean the same thing.

Claim 4: $x = 1 \Rightarrow xy = y$

- If we know $x = 1$, then we know $xy = y$.
- If we know $x \neq 1$, then we don’t know anything about $xy$.
- If we know $xy = y$, then we don’t know anything about $x$.
- If we know $xy \neq y$, then we know $x \neq 1$.

The contrapositive of Claim 4 is:
Define $P(n) : n$ is a multiple of 4, and $Q(n) : n^2$ is a multiple of 4, and consider

$$\forall n \in \mathbb{N}, P(n) \Rightarrow Q(n)$$

What do the implication, converse, and contrapositive each tell you when

- $n$ is a multiple of 4
- $n$ is not a multiple of 4
- $n^2$ is a multiple of 4
- $n^2$ is not a multiple of 4

Which do you believe, and why?
Here are some ways of expressing implication, $P \implies Q$, in English. What’s $P$ and what’s $Q$, in each case?

If nominated, I will not stand.

If you think I’m lying, then you’re a liar!

Whenever I hear that song, I think about icecream.

Differentiability is sufficient for continuity.

Matching fingerprints and a motive are enough for guilt.

You can’t stay enrolled in CSC165 without a pulse.

Successful programming requires skill.

I’ll go only if you insist.

Don’t knock it unless you’ve tried it.
vacuous truth

We’ve already separated implication from quantification, so we can make sense of

\[ P(x) \Rightarrow Q(x) \]

It’s true, except when \( P(x) \) is true and \( Q(x) \) is false. In particular, an implication is always true when the antecedent is false. For example, if your eyes wander to the consequent in

\[ \forall x \in \mathbb{R}, x^2 - 2x + 2 = 0 \Rightarrow x > x + 5 \]

...you could jump to the conclusion that the implication is false.

Vacuous truth works because there are no counterexamples. Another way of thinking about this is that the empty set is a subset of every other set.

All employees earning over 80 trillion dollars are female.
All employees earning over 80 trillion dollars are male.
All employees earning over 80 trillion dollars have mauve eyeballs and breathe ammonia.
Suppose Al quits. Now consider the statement:

Every male employee earns between 25,000 and 45,000.

Is the statement true? What about its converse?

<table>
<thead>
<tr>
<th>EMPLOYEE</th>
<th>GENDER</th>
<th>SALARY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Betty</td>
<td>female</td>
<td>500</td>
</tr>
<tr>
<td>Carlos</td>
<td>male</td>
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</table>

An employee is male if, and only if, that employee earns 25,000–45,000. This is a double implication, $P \Rightarrow Q$ and $Q \Rightarrow P$, or $P \Leftrightarrow Q$. Thought of as sets, they are equal (mutual subsets).
more equivalence

How do you feel about

\[ \forall x \in \mathbb{R}, x^2 - 2x + 2 = 0 \iff x > x + 5. \]

Break it into two implications:

\[ \forall x \in \mathbb{R}, x^2 - 2x + 2 = 0 \Rightarrow x > x + 5. \]
\[ \forall x \in \mathbb{R}, x > x + 5 \Rightarrow x^2 - 2x + 2 = 0. \]

The truth values are the same. English phrases:

P is necessary and sufficient for Q.
P is true exactly when Q is true.
P implies Q, and conversely.
You need both rigor and intuition to solve problems you haven’t seen a template for. In this course I’ll present open-ended problems, and recommend the following steps for getting started on them:

**Understand the problem:** Know what’s given, what’s required. Re-state the problem in your own words, perhaps draw some diagrams.

**Plan solution(s):** If you’ve seen something similar, you may be able to use its *result* or its *method*. Work backwards: assume you’ve solved the problem and think about the next-to-last step. Try solving simpler, smaller versions of the problem. Have more than one plan before you attack the problem (!).

**Carry out your plan:** Does it lead somewhere? If not, repeat earlier steps. Articulate *exactly* why and how you’re stuck (if you are).

**Review:** Look back to savour breakthroughs and think about roadblocks. Verify your solution as much as possible. Convince a skeptical peer that you have a solution. Extend your solution to new problems...
summary

- universal claim, $\forall -$ negation?
- existential claim, $\exists -$ negation?
- implication $P \Rightarrow Q$
  - verification?
  - antecedent - consequent
  - converse of $P \Rightarrow Q$ is $Q \Rightarrow P$
  - contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$
- vacuous truth
- equivalence