A proof communicates why and how you believe something to be true. You'll need to master two things:

1. Why do you believe the thing is true? This step is messy, creative, but then increasingly precise to identify (and then strengthen) the weak parts of your belief.

2. Write up (express) why you believe the thing is true. Each step of your written proof should be justified enough to convince a skeptical peer. If you detect a gap in your reasoning, you may have to go back to step 1.

Although I present a great deal of symbolic notation, we will accept carefully-structured, precise English prose. The structure, however, is required, and is the main topic of Chapter 4.
finding a proof of universally-quantified $\Rightarrow$

To support a proof of a universally-quantified implication $\forall x \in X, P(x) \Rightarrow Q(x)$, you usually need to use some already-proven statements and axioms (defined, or assumed, to be true for $X$). You hope to find a chain

\[
\begin{align*}
C2.0 & \quad \forall x \in X, P(x) \Rightarrow R_1(x) \\
C2.1 & \quad \forall x \in X, R_1(x) \Rightarrow R_2(x) \\
& \quad \vdots \\
C2.n & \quad \forall x \in X, R_n(x) \Rightarrow Q(x)
\end{align*}
\]

Such a chain shows in $n$ steps that $P(x) \Rightarrow Q(x)$, by transitivity.
proof outline

A slightly more flexible format will be required in this course. Each link in the chain is justified by mentioning the supporting evidence in a comment beside it. We show the portions of the argument where an assumption is in effect by using indentation. Here's what a generic proof that $\forall x \in X, P(x) \Rightarrow Q(x)$ might look like.

Assume $x \in X \neq x$ is generic, so what I prove about it applies to all of $X$

Assume $P(x)$. $\#$ Antecedent. Otherwise, $\neg P(x)$ means we get the implication for free.

Then $R_1(x) \neq$ by previous result $C2.0, \forall x \in X, P(x) \Rightarrow R_1(x)$

Then $R_2(x) \neq$ by previous result $C2.1, \forall x \in X, R_1(x) \Rightarrow R_2(x)$

$\vdots$

Then $Q(x) \neq$ by previous result $C2.n, \forall x \in X, R_n(x) \Rightarrow Q(x)$

Then $P(x) \Rightarrow Q(x)$ $\#$ I assumed antecedent, got consequent (AKA introduced $\Rightarrow$)

Then $\forall x \in X, P(x) \Rightarrow Q(x)$ $\#$ reasoning about generic $x$ works for all $x \in X$. 
Chains of antecedents consequents break up in asymmetrical ways. Use truth tables, venn diagrams, or rules for manipulating predicates to show

\[(P \Rightarrow R_1) \land (P \Rightarrow R_2) \Leftrightarrow (P \Rightarrow (R_1 \land R_2))\]

Notice that things switch when the conjunction is at the other end of the implication

\[((R_1 \Rightarrow Q) \land (R_2 \Rightarrow Q)) \Leftrightarrow ((R_1 \lor R_2) \Rightarrow Q)\]
odd example

The square of an odd number is odd. Prove:

$$\forall n \in \mathbb{N}, n \text{ odd } \Rightarrow n^2 \text{ odd}.$$
a real inequality

Prove that for every pair of non-negative real numbers \((x, y)\), if \(x\) is greater than \(y\), then the geometric mean, \(\sqrt{xy}\) is less than the arithmetic mean, \((x + y)/2\).
some directions work better

Prove that for any natural number $n$, $n^2$ odd implies that $n$ is odd.
another example

Prove

$$\forall x \in \mathbb{R}, x > 0 \Rightarrow \frac{1}{x+2} < 3$$

Do you believe the converse? Why?
Proof by contradiction

**Goal:** Prove $P \Rightarrow Q$.

Assume $\neg Q$ in order to derive a contradiction.

Then: some steps leading to a contradiction.

Then $\neg P$. contradiction, since $P$ is known to be true.

Then $Q$ since assuming $\neg Q$ leads to contradiction.

How does this method work?
contradiction example

Let \( P = \{ p \in \mathbb{N} : p \) has exactly two factors} \)

Prove by contradiction

\[ sp : \quad \forall n \in \mathbb{N}, \ |P| > n \]