1. Consider the following statements:

(S1) Programs that passed test 1 also passed test 2.
(S2) Programs passed test 2 unless they failed test 1.
(S3) Programs passed test 2 only if they passed test 1.

(a) Rewrite statements (S1), (S2), and (S3) using precise symbolic notation.

Solution:
Let \( P \) represent the set of all programs.
Let \( T_1(p) \) denote: “program \( p \) passed test 1”.
Let \( T_2(p) \) denote: “program \( p \) passed test 2”.
Then, (S1) can be written: \( \forall p \in P, T_1(p) \rightarrow T_2(p) \).
(S2) can be written: \( \forall p \in P, T_1(p) \leftrightarrow T_2(p) \).
(S3) can be written: \( \forall p \in P, T_2(p) \rightarrow T_1(p) \).

Alternate Solution: Let \( T(p, n) \) denote: “program \( p \) passed test \( n \)”. Then, in the solution above, replace every “\( T_1(p) \)” with \( T(p, 1) \) and every “\( T_2(p) \)” with \( T(p, 2) \).

(b) Which of the three statement have the same meaning?
Solution: Based on the answers for part (a), none of the three statements have the same meaning (i.e., all three have a different meaning).

2. Consider the statement:

(S4) All Java programs passed test 1.

(a) Rewrite (S4) using implication but no quantification.

Solution: “If a program was written in Java, then it passed test 1.”

(b) Rewrite (S4) using precise symbolic notation.

Solution:
Let \( P \) represent the set of all programs, \( J(p) \) denote: “program \( p \) was written in Java”, and \( T_1(p) \) denote: “program \( p \) passed test 1”.
Then, (S4) can be written: \( \forall p \in P, J(p) \rightarrow T_1(p) \).

Note: It would be reasonable but not as good to say instead:
Let \( J \) represent the set of all Java programs and \( T_1(p) \) denote: “program \( p \) passed test 1”.
Then, (S4) can be written \( \forall p \in J, T_1(p) \).
The reason why this is not as good as the first answer is because it makes it impossible to talk about programs other than the ones in \( J \). If we wanted to talk about programs not written in Java, it would not be good enough (and it would be incorrect notation) to write something like “\( \forall p \notin J \)” because this would have the meaning of “for everything that is not a Java program” (including people, birds, colours, . . .).
The first answer is the best way to allow the possibility that every property mentioned in the statement could be true or false, while limiting the domain of discussion to the type of object we are interested in.

(c) Write the contrapositive of (S4), symbolically and in English.

Solution: \( \forall p \in P, \neg T_1(p) \rightarrow \neg J(p) \).
“Programs that failed test 1 were not written in Java.”

(d) Write the converse of (S4), symbolically and in English.

Solution: \( \forall p \in P, T_1(p) \rightarrow J(p) \).
“All programs that passed test 1 were written in Java.”
3. Draw a Venn diagram with sets to represent “programs written in C”, “programs that passed test 1”, and “programs that passed test 2” (make sure that your sets overlap to divide the diagram into eight regions). Then, for each program, write the program number in the appropriate region of your diagram (based on the information in the database above).

**Solution:**

Let $P$ denote the set of all programs.
Let $C$ denote the set of programs written in C.
Let $T_1$ denote the set of programs that passed test 1.
Let $T_2$ denote the set of programs that passed test 2.
The diagram is given in the figure below (or at the top of the next page).

![Venn Diagram](image)

Figure 1: Venn diagram for Question 3.

4. Draw three copies of your diagram from the preceding question. On the first copy, shade the region(s) that corresponds to “programs that have passed every test”. On the second copy, shade the region(s) that corresponds to “programs that have passed some test”. On the third copy, shade the region(s) that corresponds to “programs that have passed no test”.

**Solution:** In the figures below or at the top of the next page, grey regions represent the “shaded” parts (white regions are “unshaded”). Also note that we take the meaning of “some” to be inclusive (i.e., a program that passed every test is considered to have passed some test).

5. State whether each statement below is true or false. When appropriate, justify your answer by citing a specific counter-example.

(S5) Every Java program passed some test.
(S6) Some Java program passed no test.
(S7) No C program passed every test.

**Solution:**

(S5) is true, from Figure ?? (or directly from the database: program 1 passed test 2, program 2 passed test 2, and program 4 passed test 1).
(S6) is false, from Figure ?? (also, because it is the negation of (S5)).
(S7) is false, from Figure ??: program 6 is a counter-example.

6. Let $P$ denote the sentence “$x$ is even”. Let $Q$ denote the sentence “$x^2$ is even”. Write the sentence $P \rightarrow Q$ in English:
Figure 2: Programs that have passed every test.

Figure 3: Programs that have passed some test.

(a) Using the words “if, then”.
(b) Using the word “implies”.
(c) Using the words “only if”.
(d) Using the words “is necessary for”.
(e) Using the words “is sufficient for”.

Solution:
(a) If $x$ is even, then $x^2$ is even.
(b) $x$ is even implies $x^2$ is even.
(c) $x$ is even only if $x^2$ is even.
(d) It is necessary that $x^2$ be even for $x$ to be even.

Note that “$x^2$ is even is necessary for $x$ is even” is not even a correct English sentence! Part of the point of this question was to make people realize that translating from symbolic notation to English involves more than just one-by-one replacement of symbols by words. You have to think about the meaning of the symbols and write a correct English sentence that has the same meaning.

(e) It is sufficient that $x$ be even for $x^2$ to be even.
7. Consider the following sentence about integers $a, b, c$:

If $a$ divides $bc$, then $a$ divides $b$ or $a$ divides $c$.

For each sentence below, state whether it is the same as, the negation of, the converse of, the contrapositive of, or unrelated to the statement above. Justify each of your answers briefly (e.g., by writing both statements in symbolic notation).

(a) If $a$ divides $b$ or $a$ divides $c$, then $a$ divides $bc$.
(b) If $a$ does not divide $b$ or $a$ does not divide $c$, then $a$ does not divide $bc$.
(c) $a$ divides $bc$ and $a$ does not divide $b$ and $a$ does not divide $c$.
(d) If $a$ does not divide $b$ and $a$ does not divide $c$, then $a$ does not divide $bc$.
(e) $a$ does not divide $bc$ or $a$ divides $b$ or $a$ divides $c$.
(f) If $a$ divides $bc$ and $a$ does not divide $c$, then $a$ divides $b$.

Solution: The standard mathematical notation used to represent “divides” is a vertical bar, so for example, “$a$ divides $bc$” can be represented symbolically as $a|bc$.

It would also be perfectly fine to introduce a predicate letter like $D(x, y)$ to represent “$x$ divides $y$” (so that “$a$ divides $bc$” becomes $D(a, bc)$, for example), or even to introduce three unrelated letters to represent each of the three sentences (e.g., $P$ to represent “$a$ divides $bc$”, etc.), since we are only concerned with the logical structure of the various sentences, not their meaning.

If we use the vertical bar notation, the original sentence can be written symbolically as follows:

$$a|bc \rightarrow a|b \lor a|c$$

(a) The sentence becomes: $a|b \lor a|c \rightarrow a|bc$.
   This is the converse of the original sentence.

(b) The sentence becomes: $¬(a|b) \lor ¬(a|c) \rightarrow ¬(a|bc)$.
   This is unrelated to the original sentence.

(c) The sentence becomes: $a|bc \land ¬(a|b) \land ¬(a|c)$.
   This is the negation of the original sentence.
(d) The sentence becomes: \( \neg(a \mid b) \land \neg(a \mid c) \rightarrow \neg(a \mid bc) \).
This is the contrapositive of the original sentence.

(e) The sentence becomes: \( \neg(a \mid bc) \lor a \mid b \lor a \mid c \).
This is equivalent to the original sentence.

(f) The sentence becomes: \( a \mid bc \land \neg(a \mid c) \rightarrow a \mid b \).
This is equivalent to the original sentence.

(g) The sentence becomes: \( a \mid bc \lor \neg(a \mid b) \rightarrow a \mid c \).
This is unrelated to the original sentence.

8. For each quantified statement below, rewrite the statement in English and state whether it is true or false. When appropriate, justify your answer with an example or counter-example.

(a) \( \exists m \in \mathbb{N} \forall n \in \mathbb{N}, m > n \)
(b) \( \forall m \in \mathbb{N} \exists n \in \mathbb{N}, m > n \)
(c) \( \forall n \in \mathbb{N} \exists m \in \mathbb{N}, m > n \)

Solution:
(a) One way to translate the statement is: “There exists a natural number larger than all other natural numbers.”
This statement is clearly false: there is no “largest” natural number.

(b) One way to translate the statement is: “For all natural numbers, there is another natural number that is smaller.”
This statement is false because there is no natural number smaller than 0 (so when \( m = 0 \), no value of \( n \) exists to satisfy \( m > n \)).

(c) One way to translate the statement is: “For all natural numbers, there is another natural number that is greater.”
This statement is true because, for example, every natural number \( n \) has a successor \( n + 1 \).

9. In calculus, a function \( f \) with domain \( \mathbb{R} \) (the real numbers) is defined to be strictly increasing provided that for all real numbers \( x \) and \( y \), \( f(x) < f(y) \) whenever \( x < y \). Complete each of the following sentences using the appropriate symbolic notation.

(a) A function \( f \) with domain \( \mathbb{R} \) is strictly increasing provided that . . .
Solution: \( \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x < y \rightarrow f(x) < f(y) \)

(b) A function \( f \) with domain \( \mathbb{R} \) is not strictly increasing provided that . . .
Solution: \( \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x < y \land f(x) \not< f(y) \)
Alternate Solution: \( \neg(\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x < y \rightarrow f(x) < f(y)) \)
This is not as good as the first solution: in general, expressing the negation of a statement simply as “\( \neg \) statement” gives little information compared to taking the time to “work the negation in”.

10. Find three sets \( A, B, C \) with as few elements as possible so that statement (S1) below is true but statement (S2) is false.

(S1) \( \forall x \in A \exists y \in B, x + y \in C \)
(S2) \( \exists y \in B \forall x \in A, x + y \in C \)

Solution: Let \( A = \{0, 1\}, B = \{0, 1\}, C = \{1\} \).
Then, (S1) is true because for \( x = 0 \), we can pick \( y = 1 \) and for \( x = 1 \), we can pick \( y = 0 \). However, (S2) is false because when \( y = 0, x = 0 \) fails \( x + y \not\in C \) and when \( y = 1, x = 1 \) fails.
11. At a murder trial, four witnesses give the following testimony.

**Alice:** If either Bob or Carol is innocent, then so am I.

**Bob:** Alice is guilty, and either Carol or Dan is guilty.

**Carol:** If Bob is innocent, then Dan is guilty.

**Dan:** If Bob is guilty, then Carol is innocent; however, Bob is innocent.

(a) Is the testimony consistent, i.e., is it possible that everyone is telling the truth?

**Solution:** First, we define some abbreviations:

- **A:** “Alice is innocent”
- **B:** “Bob is innocent”
- **C:** “Carol is innocent”
- **D:** “Dan is innocent”

Then, we can rewrite the statements symbolically (with the understanding that “guilty = ¬ innocent”):

**Alice:** \((B \lor C) \rightarrow \neg A\)

**Bob:** \(\neg A \land (\neg C \lor \neg D)\)

**Carol:** \(B \rightarrow \neg D\)

**Dan:** \((\neg B \rightarrow C) \land B\)

Now, let us assume that everyone’s testimony is true. We will see that this leads to a contradiction, which means that someone must be lying.

Bob’s testimony means that Alice is guilty. Then, the contrapositive of Alice’s testimony means that both Bob and Carol are guilty. But this contradicts Dan’s testimony that Bob is innocent.

(b) If every innocent (not guilty) person tells the truth and every guilty person lies, determine (if possible) who is guilty and who is innocent.

**Solution:** Using the same notation as above, there are many ways of determining who is innocent and who is guilty, including trial-and-error. One possibility is that Alice and Carol are guilty while Bob and Dan are innocent. Then, Alice’s statement is false (because \(B\) is true but \(A\) is false), Bob’s statement is true (because \(A\) is false and \(C\) is false), Carol’s statement is false (because \(B\) is true and \(D\) is true), and Dan’s statement is true (because \(B\) is true), as required.

12. Consider the following statement: (1)

“If a program has a syntax error, then the program will not compile.”

(a) Define the domain and predicates necessary to translate the statement into precise symbolic notation.

**Domain:** The set of programs, \(P\).

Let \(S(x)\) represent \(x\) has a syntax error.

Let \(C(x)\) represent \(x\) will compile.

(b) Translate (1) into precise symbolic notation. \(\forall p \in P, S(p) \rightarrow \neg C(p)\)

(c) (4 marks) Give the converse of (1) first in English, then in precise symbolic notation.

If a program does not compile, then it has a syntax error.

\(\forall p \in P, \neg C(p) \rightarrow S(p)\)

(d) Give the contrapositive of (1) first in English, then in precise symbolic notation.

If a program compiles then it does not have a syntax error.

\(\forall p \in P, C(p) \rightarrow \neg S(p)\)
(e) (4 marks) Give the contrapositive of your answer to ?? in precise symbolic notation.

\[ \forall p, \neg S(p) \rightarrow C(p) \]  

i.e., in words:

If a program does not have a syntax error then it will compile.

13. (40 marks) Assume you are given the following predicate symbols and your domain is \( \mathbb{N} \), the set of natural numbers (we assume that \( 0 \in \mathbb{N} \)).

\( g(x, y) \): \( x \) is greater than \( y \)
\( e(x, y) \): \( x \) equals \( y \)
\( \text{sum}(x, y, z) \): \( x + y = z \)
\( \text{prod}(x, y, z) \): \( x \cdot y = z \)

Translate the following statements into idiomatic English:

(a) \( \forall x \in \mathbb{N}, g(x, 0) \)

Every natural number is greater than 0.

(b) \( \forall x \in \mathbb{N}, \exists z \in \mathbb{N}, \text{prod}(z, 2, x) \)

Every natural number is even.

(c) \( \forall a \in \mathbb{N}, \forall b \in \mathbb{N}, \forall c \in \mathbb{N}, (g(a, b) \land g(b, c)) \rightarrow g(a, c) \)

For all natural numbers, if \( a \) is greater than \( b \) and \( b \) greater than \( c \), then \( a \) is greater than \( c \).

(d) \( \neg (\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, g(m, n)) \)

There does not exist a largest number

We get this from first doing a literal translation;

It is not the case that, for any natural number there exists a greater number.

Translate the following English statements into precise symbolic notation. Only use the predicates and domain defined above. Make sure you specify the domain of your variables in your solution and that your predicates are boolean.

(e) Every positive multiple of 5 is greater than 7.

\( \forall x \in \mathbb{N} \exists y \in \mathbb{N}, p(y, 5, x) \rightarrow g(x, 7) \)

(f) There is an odd integer.

\( \exists x \in \mathbb{N} \exists y \in \mathbb{N}, \exists z \in \mathbb{N}, \text{prod}(y, 2, z) \land \text{sum}(z, 1, x) \)

(g) If \( x + y = z \) then \( y + x = z \).

\( \forall x \in \mathbb{N}, \forall y \in \mathbb{N}, \forall z \in \mathbb{N}, \text{sum}(x, y, z) \rightarrow \text{sum}(y, x, z) \)

(h) Not all integers are multiples of 2.

\( \neg \forall x \in \mathbb{N}, \exists y \in \mathbb{N}, \text{prod}(y, 2, x) \)

14. Logic Puzzle: There are many brain teasers involving deserted islands and the people who inhabit them. One such puzzle, involves an island consisting of two different races. The Truth Tellers and the Liars. The Truth Tellers always tell the truth and the Liars, falsehoods. Suppose you meet three people \( U, V \) and \( W \) from this island. The first person \( U \) does not speak your language however \( V \) offers to translate. For each case, determine (if possible) from their responses to the following question, which race they each belong. If it is not possible, clearly show why it is not possible to determine which race at least one of \( U, V \) or \( W \) belong to.

How many of you are Truth Tellers?
Responses:

(a) V: “U said, ‘Exactly one of us is a Truth Teller.’”
W: “Don’t believe V. He is a Liar.”
If V is a Truth Teller then U must have said Exactly one of us is a Truth Teller. This implies that U must be a Liar and so must W. This coincides with what W says, so V is a Truth Teller and U and V are Liars.

(b) V: “U said, ‘Exactly one of us is a Liar.’”
W: “V’s statement is true.”
If V is telling the truth and U said Exactly one of us is a Liar then both V and W are Truth Tellers. However, that means that U must be a Liar, which means that U told the truth and therefore cannot be a Liar. Therefore we know that V cannot have told the truth.
Suppose that V is a Liar. Then U could not have said Exactly one of us is a Liar and W’s statement is a lie. Therefore W and V are liars. What about U? We don’t know what U said, so we don’t know whether U is a liar or not. Therefore we can not determine which race U belongs to.

15. Consider our example from class about rainy days.

(2) Every rainy day I bring an umbrella.
(3) If I bring an umbrella, then I stay dry.

(a) For each of the following statements, determine whether it has the same meaning as (2). If it has a different meaning, make a small alteration to the statement so that it has the same meaning.

i. I bring an umbrella, if it is a rainy day. same
ii. If it is a rainy day, I bring an umbrella. same
iii. I bring an umbrella only if it is a rainy day. It is a rainy day only if I bring an umbrella.
iv. A rainy day is sufficient for me to bring an umbrella. same
v. A rainy day is necessary for me to bring an umbrella. For me to bring an umbrella it is necessary for it to be a rainy day.

(b) Assume that it is a rainy day. What conclusions can you draw given statements (2) and (3). Explain your reasoning.
If it is a rainy day, then I will bring an umbrella and since I have an umbrella, I will stay dry. Therefore I can conclude that I stay dry and have an umbrella.

(c) Now assume that I forgot my umbrella. What conclusions can you draw? Explain your reasoning.
If I don’t have an umbrella, then it cannot be rainy since otherwise statement (2) would be false. However, I may or may not be dry because statement (3) is of the form false → p and is therefore true no matter what the truth value of p.

16. Recall the table of hockey stats from class.

<table>
<thead>
<tr>
<th>Number</th>
<th>Pos.</th>
<th>Player</th>
<th>Team</th>
<th>GP</th>
<th>G</th>
<th>A</th>
<th>PTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>C</td>
<td>Alexei Zhamnov</td>
<td>PHI</td>
<td>11</td>
<td>4</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>RW</td>
<td>Jarome Iginla</td>
<td>CAL</td>
<td>13</td>
<td>6</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>Joe Sakic</td>
<td>COL</td>
<td>11</td>
<td>7</td>
<td>5</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>C</td>
<td>Vincent Damphousse</td>
<td>SJ</td>
<td>11</td>
<td>5</td>
<td>6</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>RW</td>
<td>Martin St. Louis</td>
<td>TB</td>
<td>9</td>
<td>4</td>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>6</td>
<td>LW</td>
<td>Fredrik Modin</td>
<td>TB</td>
<td>9</td>
<td>5</td>
<td>6</td>
<td>11</td>
</tr>
<tr>
<td>7</td>
<td>C</td>
<td>Saku Koivu</td>
<td>MON</td>
<td>11</td>
<td>3</td>
<td>8</td>
<td>11</td>
</tr>
<tr>
<td>8</td>
<td>C</td>
<td>Peter Forsberg</td>
<td>COL</td>
<td>11</td>
<td>4</td>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>9</td>
<td>RW</td>
<td>Alexei Kovalev</td>
<td>MON</td>
<td>11</td>
<td>6</td>
<td>4</td>
<td>10</td>
</tr>
</tbody>
</table>
G5: players who have scored more than 5 goals

GP11: players who have played more than 11 games

M: players who have scored more points than games played

(a) Draw a Venn diagram with sets that show “players with 5 or more goals”, “players who have played at least 11 games”, “players who have more points than games played”. Using the information in the table, enter each player’s number into the appropriate region of the diagram.

(b) Make a copy of the diagram in (a) and shade in the region that corresponds to the statement “All players who have played less than 11 games yet scored more points than games played”.

(c) Make another copy of the diagram in (a) and shade in the region that corresponds to the statement “Every player that has more points than games played and has scored at least 5 goals”.

17. The following was heard on TV recently:

“Product X is so good, it won an award!”

The makers of product X want you to believe certain (possibly implicit) hypothesis/hypotheses and conclusion(s).

(a) Write these down explicitly.

(b) Formalize them with appropriate domains and predicates.

18. There is a set P, of problems that are “polynomial time solvable” and a set NP, of problems that are only “non-deterministically polynomial time solvable”. Consider the following statement:

(S1) “If \( P = NP \), then the problem SAT is polynomial time solvable.”
(a) Define the domain and predicates necessary to translate the statement into precise symbolic notation.
(b) Translate (S1) into precise symbolic notation.
(c) Give the converse of (S1) first in English, then in precise symbolic notation.
(d) Give the contrapositive of (S1) first in English, then in precise symbolic notation.
(e) Give the contrapositive of your answer to ?? in precise symbolic notation.

Now consider the same sentence expressed using quantification:
(S2) If every problem is polynomial time solvable if and only if it is non-deterministically polynomial time solvable, then the problem SAT is polynomial time solvable.”

(f) Define the domain and predicates necessary to translate the statement into precise symbolic notation.
(g) Translate (S2) into precise symbolic notation. Simplify your answer such that only predicates are negated (not entire sentences).
(h) Give the converse of (S2) first in English, then in precise symbolic notation. Simplify your answer such that only predicates are negated.
(i) Give the contrapositive of (S2) first in English, then in precise symbolic notation. Simplify your answer such that only predicates are negated.
(j) Give the contrapositive of your answer to ??h in precise symbolic notation. Simplify your answer such that only predicates are negated.

19. Consider the following database $D$ of programs that test inputs. A program in this database may return a certificate when the input to the algorithm is accepted, rejected or both, and may or may not be linear in running time.

<table>
<thead>
<tr>
<th>Program</th>
<th>Certificate</th>
<th>Linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>reject</td>
<td>yes</td>
</tr>
<tr>
<td>2</td>
<td>accept</td>
<td>no</td>
</tr>
<tr>
<td>3</td>
<td>accept</td>
<td>yes</td>
</tr>
<tr>
<td>4</td>
<td>reject</td>
<td>no</td>
</tr>
<tr>
<td>5</td>
<td>both</td>
<td>no</td>
</tr>
<tr>
<td>6</td>
<td>neither</td>
<td>yes</td>
</tr>
</tbody>
</table>

(a) Draw a Venn diagram with a region for the programs that return a certificate when the input is accepted, a region for the programs that return a certificate when the input is rejected and a region for the programs that are linear. Add programs 1 to 6 into your diagram.
(b) Define the following predicates:

- \( A(x) \) represents “program \( x \) returns a certificate if the input is accepted.”
- \( R(x) \) represents “program \( x \) returns a certificate if the input is rejected.”
- \( L(x) \) represents “program \( x \) runs in linear time.”

For each statement redraw your Venn diagram from part (a) and shade in the region which makes the statement false.

(S1) \( A(x) \rightarrow L(x) \)

(S2) \( \neg L(x) \rightarrow A(x) \)

(c) Are the following statements true? If yes, explain why, if not, give a counterexample.

(S3) \( \forall x \in D, L(x) \rightarrow \neg(A(x) \land R(x)) \)

(S4) Some program is linear and returns a certificate on rejected input.

(S5) \( \forall x \in D, (A(x) \land R(x)) \rightarrow \neg L(x) \).

(d) Rewrite (S4) using “every” instead of “some” such that the meaning remains the same.

20. Now that we can write statements precisely, we can draw logical conclusions from a set of statements and prove that the conclusion is a consequence of the statements. These questions will help prepare you for learning to write proofs. For each set of statements, define a domain and set of predicates. Rewrite the statements and
conclusions in precise symbolic notation. Assuming that the statements are true, determine which one of the possible conclusions can be drawn from the statements. Justify your choice of conclusion by explaining how the two statements imply the conclusion.

(a) (S1) “All politicians are powerful people.”
(S2) “No powerful people are easily forgotten.”
Possible conclusions:

i. People who are easily forgotten are politicians.
ii. Politicians are not easily forgotten.
iii. No powerful people are politicians.
iv. Some easily forgotten people are politicians.

Conclusion ii). Since (S1) says that all politicians are powerful and (S2) says that powerful people are never forgotten, we can conclude that politicians are never forgotten.

(b) (S3) “If Bart gets in trouble then either Homer or Milhouse get in trouble.”
(S4) “Homer does not get in trouble.
Possible conclusions:

i. If Milhouse gets in trouble then Bart gets in trouble.
ii. If Homer does not get in trouble then Milhouse does not get in trouble.
iii. Milhouse gets in trouble if Bart gets in trouble.
iv. Bart gets in trouble whenever Homer gets in trouble.
v. Either Bart gets in trouble or Milhouse gets in trouble.

21. Assume you are given the following predicate symbols and your domain is \( \mathbb{N} \), the set of natural numbers (we assume that 0 \( \in \mathbb{N} \)).

\[
g(x, y): x \text{ is greater than } y
\]
\[
e(x, y): x \text{ equals } y
\]
\[
\text{sum}(x, y, z): x + y = z
\]
\[
\text{prod}(x, y, z): x \cdot y = z
\]

Each of the following statements is a mathematical property of the natural numbers. Translate the following statements into English and state the property.

(a) \( \forall x \in \mathbb{N}, \exists y \in \mathbb{N}, \text{prod}(x, y, x) \).
(b) \( \forall x \in \mathbb{N}, \exists y \in \mathbb{N}, \text{sum}(x, y, x) \).
(c) \( \forall x \in \mathbb{N}, \forall y \in \mathbb{N}, (\text{sum}(x, y, z) \leftrightarrow \text{sum}(y, x, z)) \)
(d) $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, \forall z \in \mathbb{N}, g(x, y) \land g(y, z) \rightarrow g(x, z)$

(e) $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, \forall z \in \mathbb{N}, \forall t \in \mathbb{N}, [\text{sum}(x, y, z) \land \text{prod}(z, t, w) \leftrightarrow (\exists u \in \mathbb{N}, \exists v \in \mathbb{N}, \text{prod}(x, t, u) \land \text{prod}(y, t, v) \land \text{sum}(u, v, w))]$

Now consider the following statements. Using the above predicates rewrite each statement in precise symbolic notation.

(f) $x$ divides $m$. Recall that divides means that $xy = m$ for some number $y$.

(g) $m$ is the smallest number that $x$ divides.

(h) $\text{LCM}(x, y, m) : m$ is the smallest number that both $x$ and $y$ divide. [Hint: Do you need to quantify the variables $x$, $y$ and $m$?]

(i) $\text{GCD}(a, b, c)$: $a$ is the greatest common divisor of $b$ and $c$. 
22. Consider the following sentence:

(S1) If \( m = 2^n - 1 \) is a prime number, then \( n \) is prime.

[Note: these types of prime numbers are called Mersenne Primes.]

Rewrite (S1) without using “If . . . , then . . .” but using:

(a) “implies”
(b) “is sufficient for”
(c) “is necessary for”
(d) “whenever”
(e) “only if”
(f) “requires”

23. Determine whether \( \exists \) can be factored from an implication. In other words, is

\[
\exists x \in X, (p(x) \rightarrow q(x)) \iff (\exists x \in X, p(x)) \rightarrow (\exists x \in X, q(x))
\]

true? Explain your reasoning. Marks will only be given for your explanation.

24. For each set of sentences, define the domain \( X \), the value of \( a \in X \) (for part b), and the predicates \( A(x) \) and \( B(x) \) such that the last sentence is false and the other sentences are true.

(a)

\[
\begin{align*}
(T) & \quad \forall x \in X, A(x) \rightarrow B(x) \\
(F) & \quad \exists x \in X, A(x) \land B(x)
\end{align*}
\]

(b)

\[
\begin{align*}
(T) & \quad \forall x \in X, A(x) \rightarrow B(x) \\
(T) & \quad \neg A(a) \\
(F) & \quad \neg B(a)
\end{align*}
\]

25. (10 marks) Let \( p(n) \) and \( q(n) \) represent the following predicates:

\[
\begin{align*}
p(n) : n \text{ is odd} & \quad q(n) : n^2 \text{ is odd}
\end{align*}
\]

where the domain is the set of integers. Determine which of the following statements are logically equivalent to each other.

(a) If the square of any integer is odd, then the integer is odd.
(b) \( \forall n \in \mathbb{Z}, (p(n) \text{ is necessary for } q(n)) \).

(c) The square of any odd integer is odd.

(d) There are some integers whose squares are odd.

(e) Given any integer whose square is odd, that integer is likewise odd.

(f) \( \forall n \in \mathbb{Z}, \neg p(n) \rightarrow \neg q(n) \).

(g) Every integer with an odd square is odd.

(h) Every integer with an even square is even.

(i) \( \forall n \in \mathbb{Z}, p(n) \) is sufficient for \( q(n) \).

26. (a) Determine whether \( \exists \) can be factored from an implication. In other words, is

\[
\exists x \in X, (p(x) \rightarrow q(x)) \iff (\exists x \in X, p(x)) \rightarrow (\exists x \in X, q(x))
\]

true?

(b) Consider

\( \forall y \in D, \exists x \in D, p(x, y) \rightarrow \exists x \in D, \forall y \in D, p(x, y) \).

i. Define \( D \) and \( p(x, y) \) such that the statement is true.

ii. Define \( D \) and \( p(x, y) \) such that the statement is false.

(c) Consider the following statement:

\[
(\forall x \in X, p(x) \rightarrow \neg \forall x \in X, q(x)) \iff (\exists x \in X, \forall y \in X (p(x) \rightarrow \neg q(y)))
\]

**Hint:** You may need to use the logical equivalence laws defined in class to simplify the statement first. If so, show and justify each step.

i. Define \( D, p(x) \) and \( q(y) \) such that the statement is true.

ii. Define \( D, p(x) \) and \( q(y) \) such that the statement is false.

27. (15 marks) Consider the statement

(1) A number is rational if it can be written as \( \frac{a}{b} \) where \( a \) and \( b \) are integers.

(a) Define a domain and a set of predicates and rewrite the statement in precise symbolic notation.

(b) Rewrite (1) using

- “sufficient”
- “only if”
- “is necessary”
- conjunction instead of implication.

(c) Write the converse and the contrapositive of the converse of (1). Is the converse true? If so, how can we alter (1) to reflect this. If not, give a counter example.