Random θ(log(n))-CNF formulas are Hard for Cutting Planes

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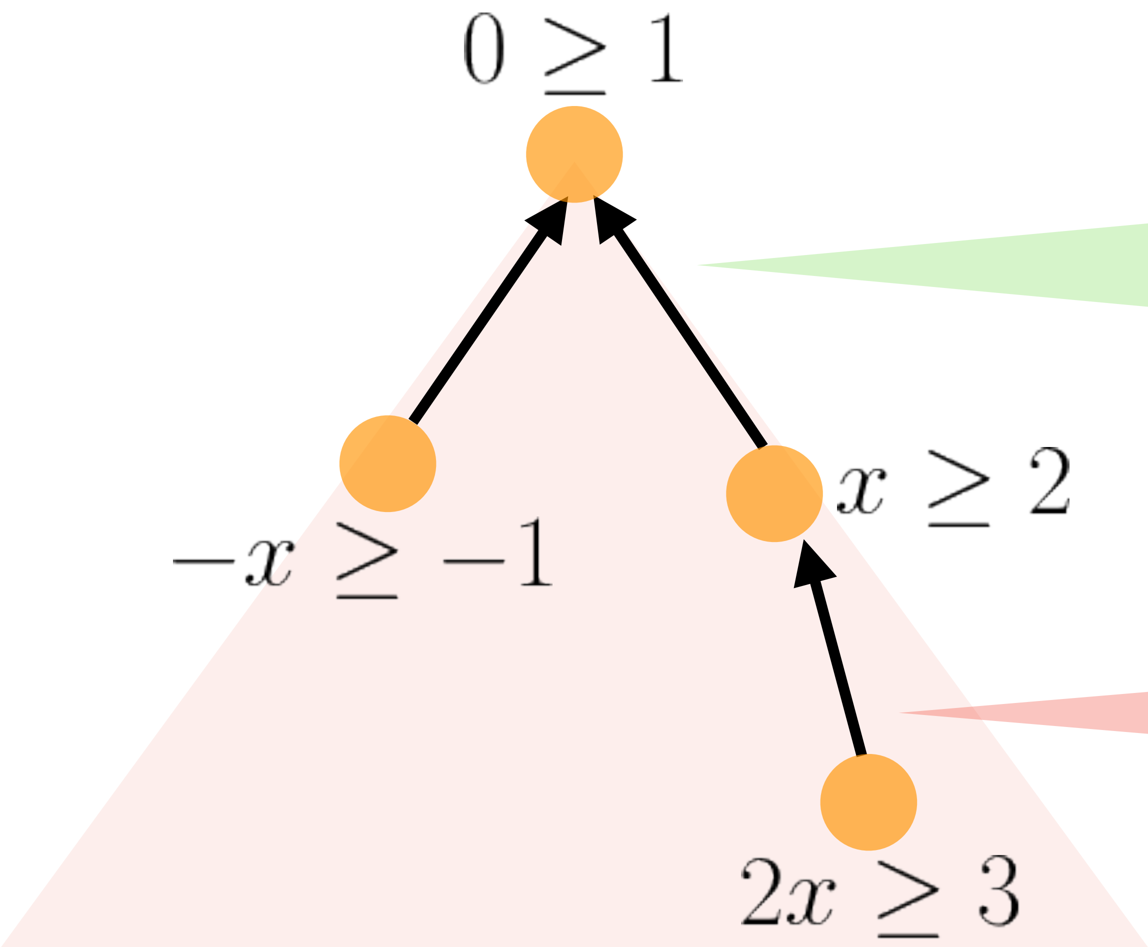
Cutting Planes



Rules

1) Addition and Multiplication by positive constant

$$\frac{Ax \geq a \qquad Bx \geq b}{c(A+B)x \geq c(a+b)} \text{ for } c \geq 0$$

2) Division with Rounding

$$\frac{dAx \geq a}{Ax \geq \lceil \frac{a}{d} \rceil}$$

Cutting Planes

- Introduced as a method of solving integer linear programming problems [Gomory63, Chvátal73]
- Has short refutations of pigeonhole principle

Feasible interpolation: For any split formula $\mathcal{F}$,

CP-Refutation of $\mathcal{F}$ $\Longrightarrow$ Real Monotone Circuit Computing a related partial function

Split Formula: $A(x, y) \wedge B(y, z)$ on variable sets x, y, z

Example: $\mathrm{Clique}(x, y) \wedge \mathrm{Coloring}(y, z)$

[Pudlak97] Cutting Planes requires an exponential number of lines to refute $\mathrm{Clique}(x, y) \wedge \mathrm{Coloring}(y, z)$.

Random SAT

Random K-CNF:

$\mathcal{F} \sim \mathcal{F}(m, n, k)$: Choose m clauses of width k uniformly at random with replacement from all possible $\binom{n}{k} 2^k$ such clauses

Clause Density $\Delta = m/n$

- Controls whether CNF is satisfiable

Threshold Conjecture: There exists a constant c_k such that for $\mathcal{F} \sim \mathcal{F}(m, n, k)$,

- if $\Delta < c_k$ then $\mathcal{F}$ is satisfiable w.h.p.,
- if $\Delta > c_k$ then $\mathcal{F}$ is unsatisfiable w.h.p.

[Ding,Sly,Sun 15] Resolved for large k

Random SAT

Random K-CNF:

$\mathcal{F} \sim \mathcal{F}(m, n, k)$: Choose m clauses of width k uniformly at random with replacement from all possible $\binom{n}{k} 2^k$ such clauses

- Testbed of hard examples for algorithms in SAT and AI

[Chvátal-Szemerédi]: Random k-CNF formulas $\mathcal{F} \sim \mathcal{F}(m, n, k)$ are w.h.p. hard for Resolution for all $k \geq 3$.

- No efficient Resolution-based algorithms for certifying unsatisfiability of random k-CNF w.h.p.

What about Cutting Planes?

Main Result

$\mathcal{F} \sim \mathcal{F}(m, n, k)$: Choose m clauses of width k uniformly at random with replacement from all possible $\binom{n}{k} 2^k$ such clauses

Theorem: Let $m = n^2 2^k$, $k = \theta(\log n)$ and sample $\mathcal{F} \sim \mathcal{F}(m, n, k)$. With high probability, any Cutting Planes refutation of $\mathcal{F}$ requires $2^{\Omega(n/\log n)}$ lines.

Proved independently by Pavel Pudlák and Pavel Hrubeš

Strategy

Feasible Interpolation: Reduces Cutting Planes refutations of *split formula* to real monotone circuits.

Strategy: Generalize feasible interpolation to work for any unsatisfiable CNF

$\mathcal{F}$: Split formula

CP-Refutation of $\mathcal{F} \Longrightarrow$ Real Monotone Circuit Computing a related partial function

[Pudlak97]

Strategy

Feasible Interpolation: Reduces Cutting Planes refutations of *split formula* to monotone circuits.

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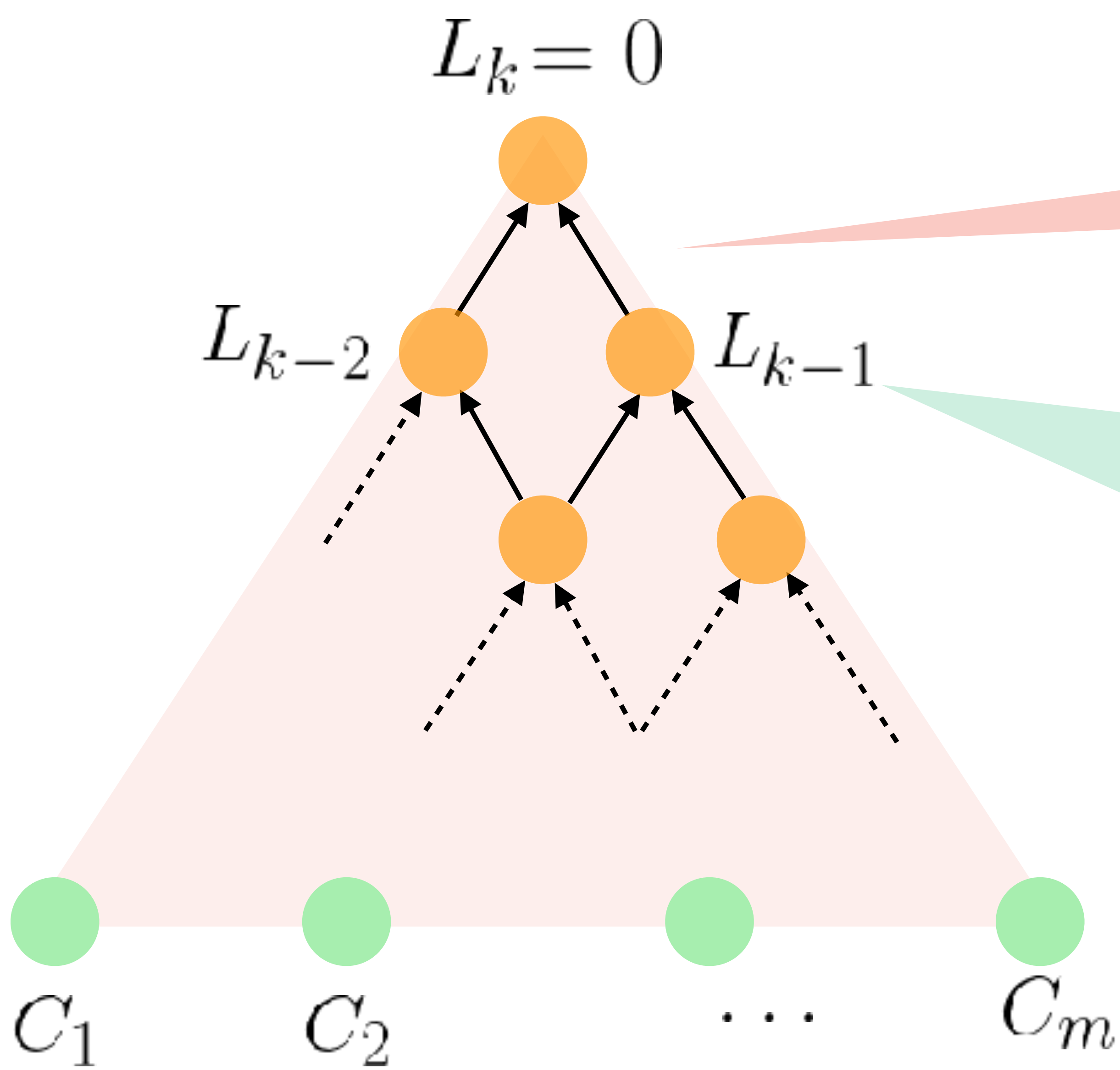
$\mathcal{F}$: Any unsatisfiable CNF

CC-Refutation of $\mathcal{F} \Longleftrightarrow$ Monotone Circuit Computing a related partial function

CC Refutations

Unsatisfiable $\mathcal{F}(X,Y) = C_1(x,y) \wedge \ldots \wedge C_m(x,y)$ over partition $X \cup Y$



Inference Rules:

Any Sound Inference

Lines:

$$L_i : \{0,1\}^n \to \{0,1\}$$

such that L_i has a small communication protocol over partition $X \cup Y$

Size: Number of lines

Strategy

CC-Refutation of $\mathcal{F}(X, Y)$

↓

Communication protocol
for the search problem

↓

Communication protocol
for the Karchmer-Wigderson game

↓

Monotone circuit
separating minterms from maxterms

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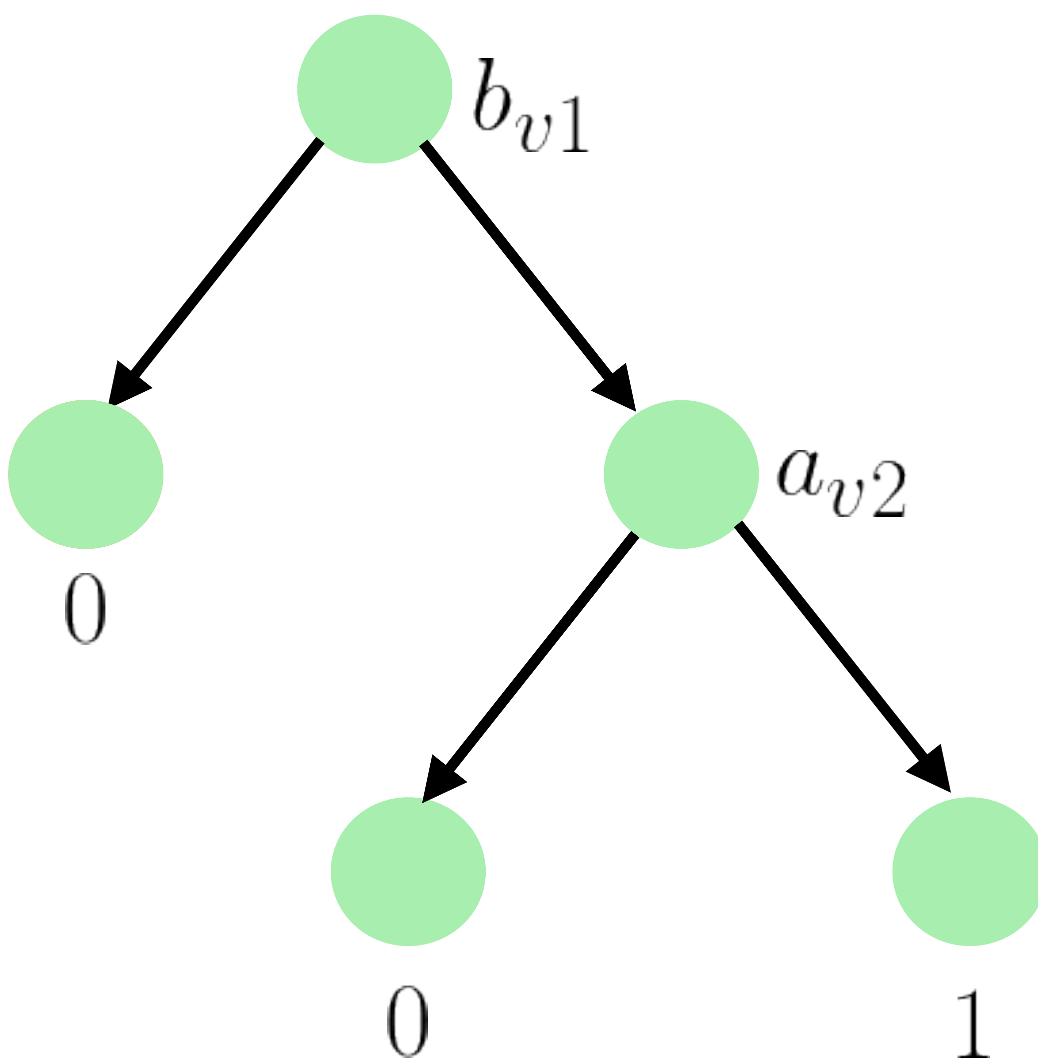
Intuition for DAG-like Protocol

Deterministic CC Protocol
computing $f(x, y)$

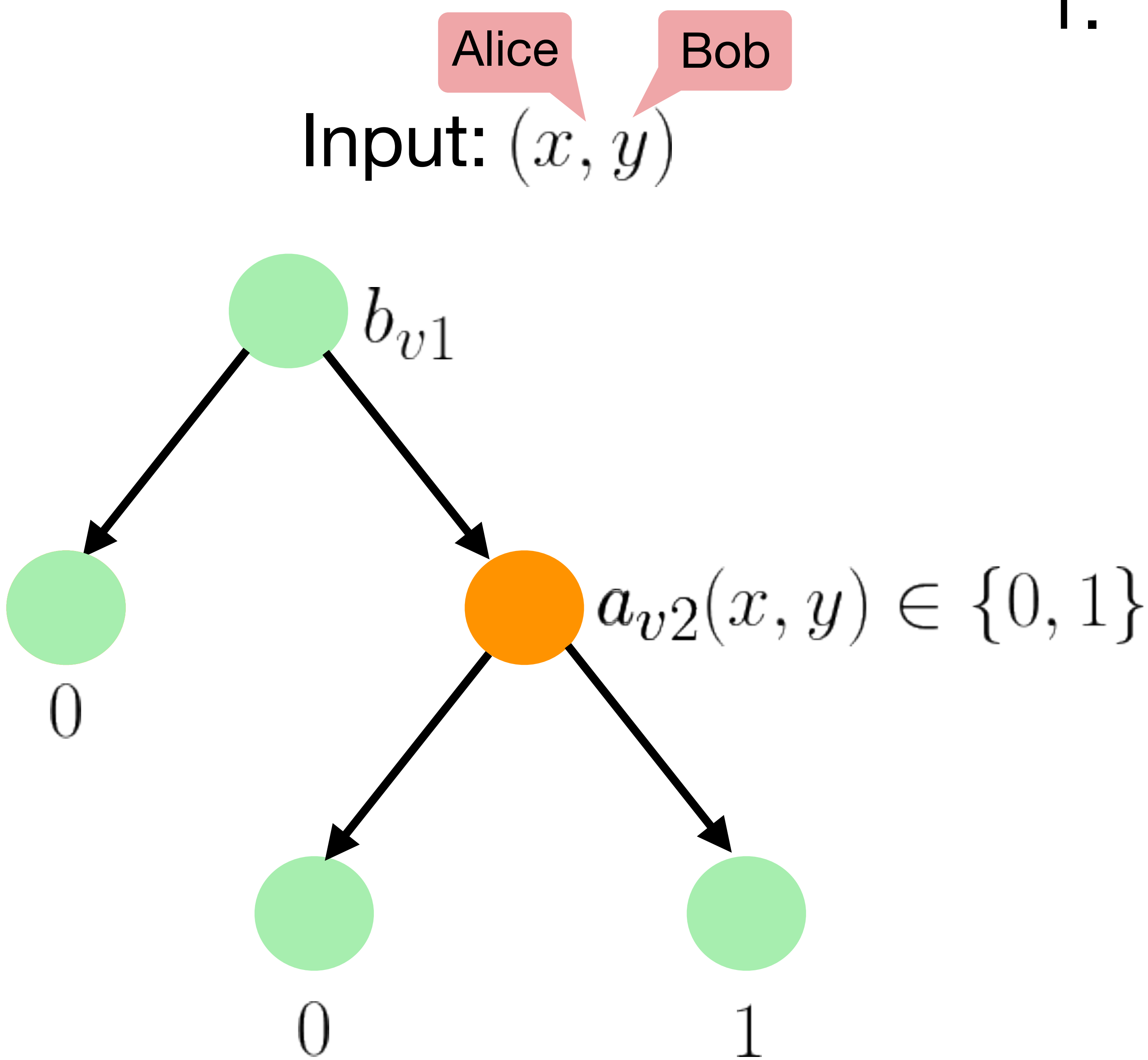
Alice Bob

Input: (x, y)



Intuition for DAG-like Protocol

Deterministic CC Protocol computing $f(x, y)$

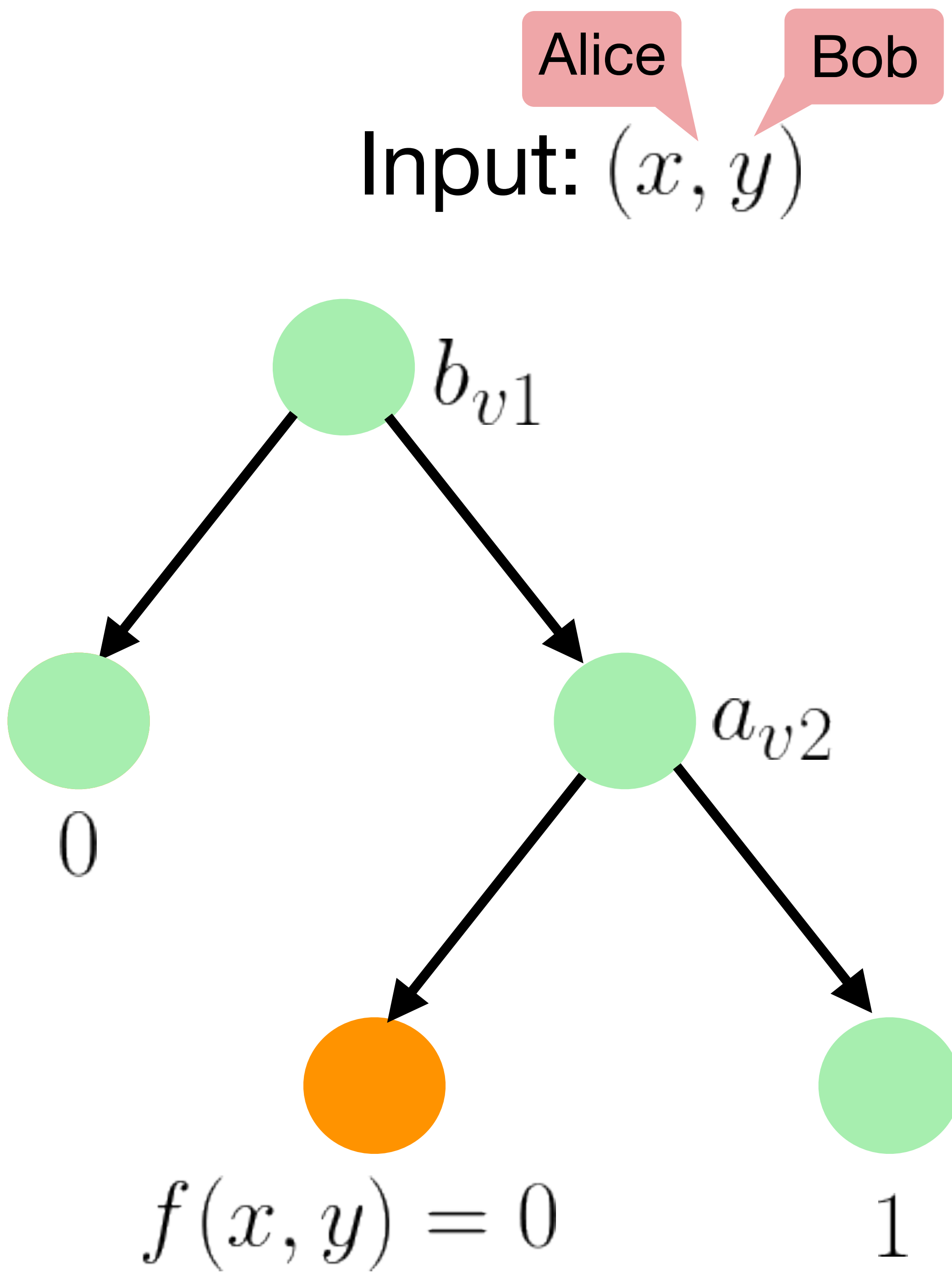


Properties

1. Non-leaf: exactly one child consistent with (x, y), players can efficiently determine which.

Intuition for DAG-like Protocol

Deterministic CC Protocol computing $f(x, y)$



Properties

1. Non-leaf: exactly one child consistent with (x, y), players can efficiently determine which
2. Leaf: labelled with α s.t. $f(x, y) = \alpha$

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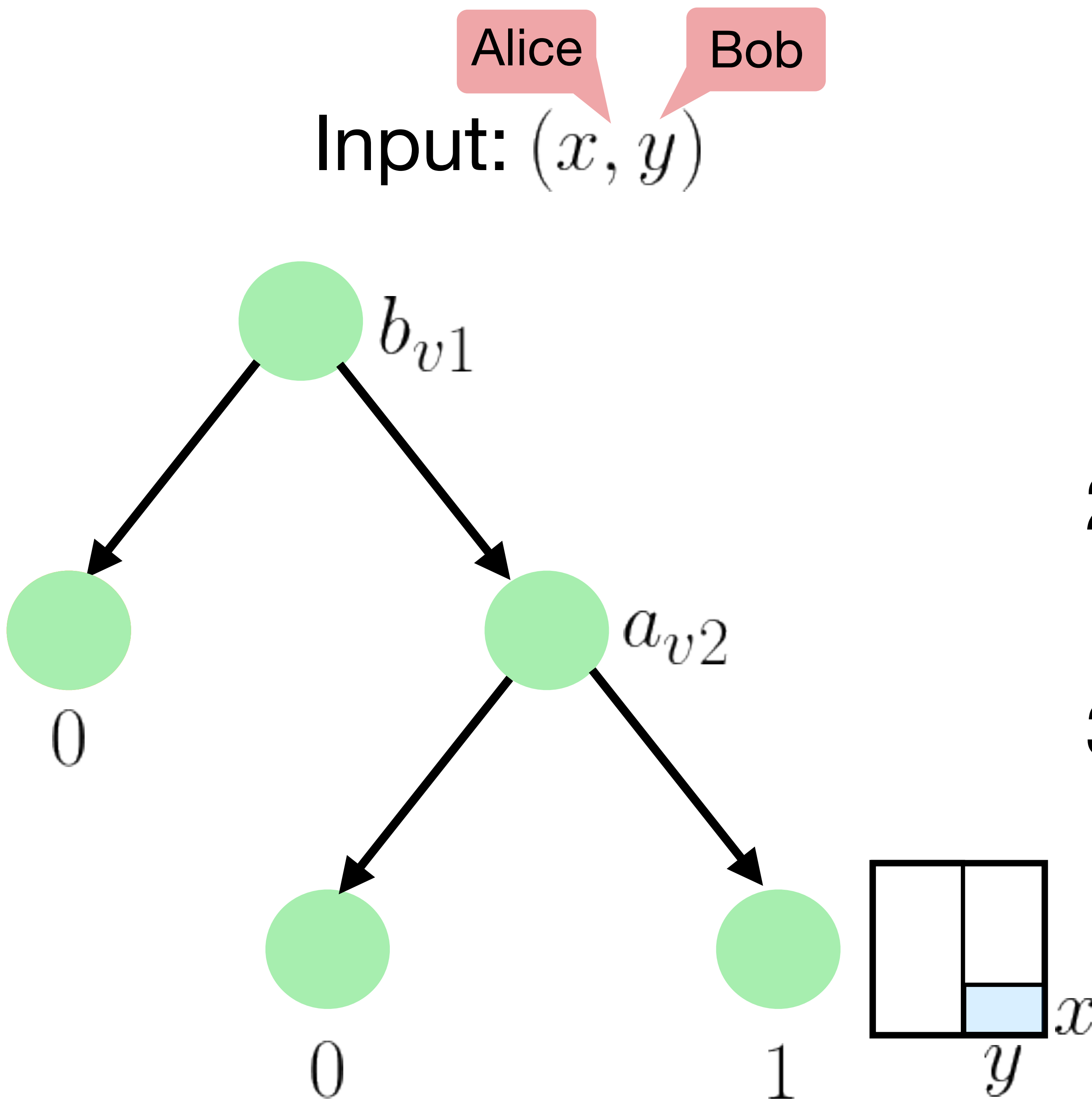


Properties

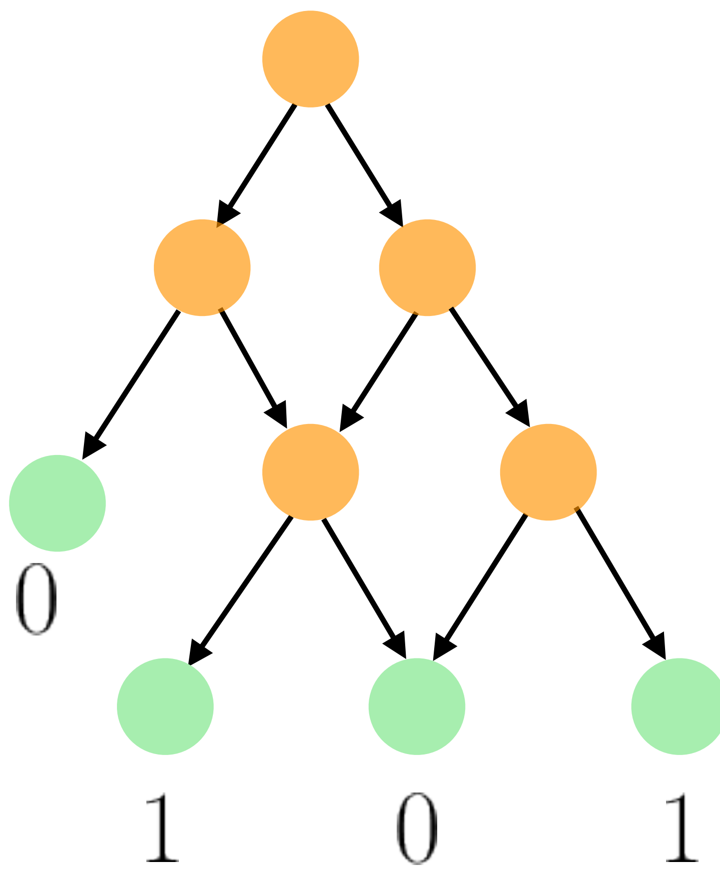
1. Non-leaf: exactly one child consistent with (x, y), players can efficiently determine which
2. Leaf: labelled with α s.t. $f(x, y) = \alpha$
3. For every node, players can efficiently check if they can reach this node on input (x, y)

CC-Games (PLS games [Razborov95])

CC-Game Computing $f(x, y)$

Alice Bob

Input: (x, y)



Satisfying:

1. Non-leaf: exactly one child consistent with (x, y), players can efficiently determine which
2. Leaf: labelled with α s.t. $f(x, y) = \alpha$
3. For every node, players can efficiently check if they can reach this node on input (x, y)

Strategy

CC-Refutation of $\mathcal{F}(X, Y)$

Communication protocol
for the search problem

Communication protocol
for the Karchmer-Wigderson game

Monotone circuit
separating minterms from maxterms

Strategy

CC-Refutation of $\mathcal{F}(X, Y)$

CC-game
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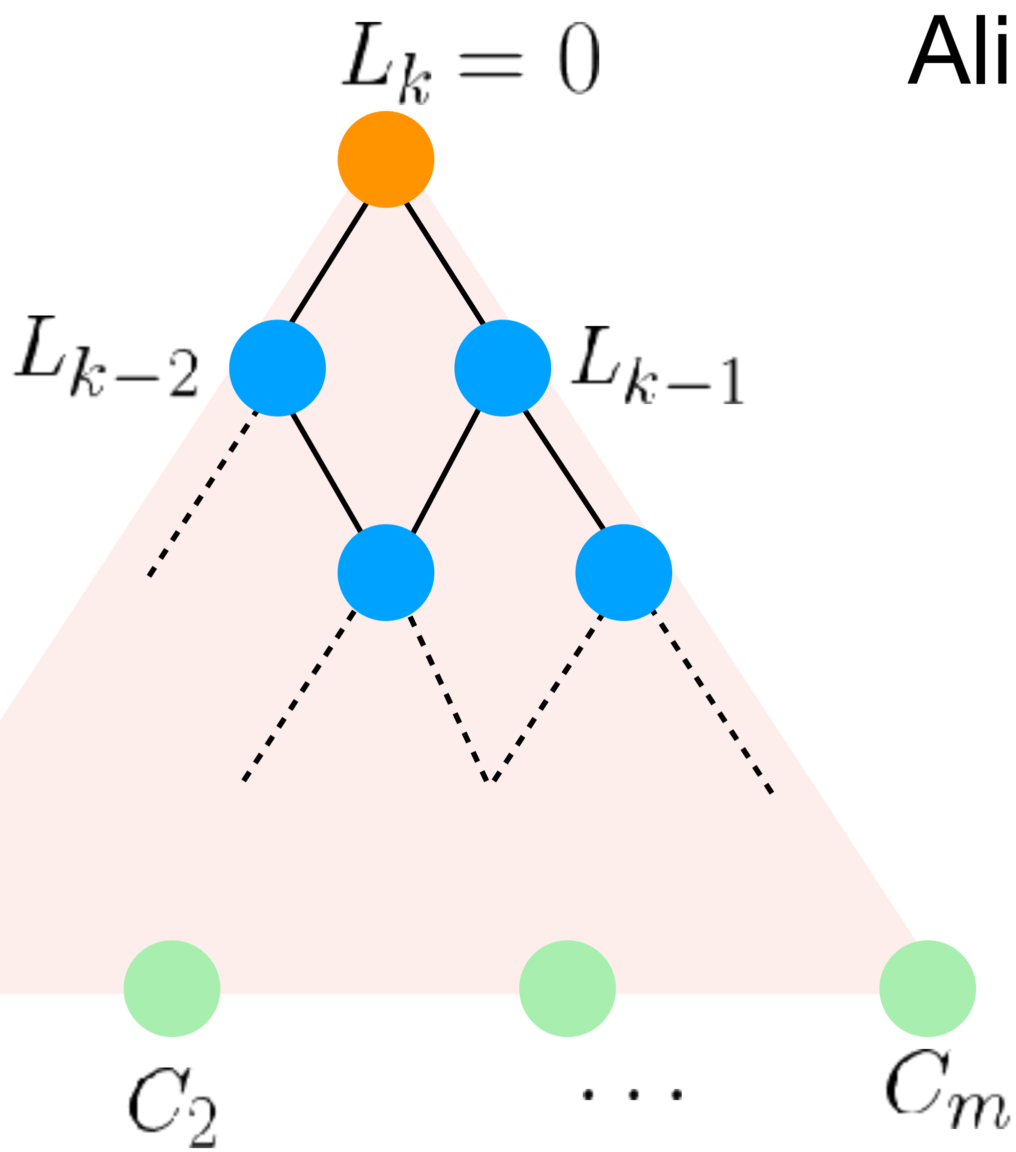
CC-Refutation $\mathcal{F}(X,Y)$ ⇨ Protocol $\mathrm{Search}_{X,Y}(\mathcal{F})$

Unsatisfiable $\mathcal{F}(X,Y) = C_1(x,y) \wedge \ldots \wedge C_m(x,y)$ over partition $X \cup Y$

$\mathrm{Search}_{X,Y}(\mathcal{F})$: Given truth assignment (x,y), output $i \in [m]$ such that $C_i(x,y) = 0$

CC-refutation of $\mathcal{F}$:

Alice: x Bob: y



CC-Refutation $\mathcal{F}(X,Y) \Rightarrow$ Protocol $\mathrm{Search}_{X,Y}(\mathcal{F})$

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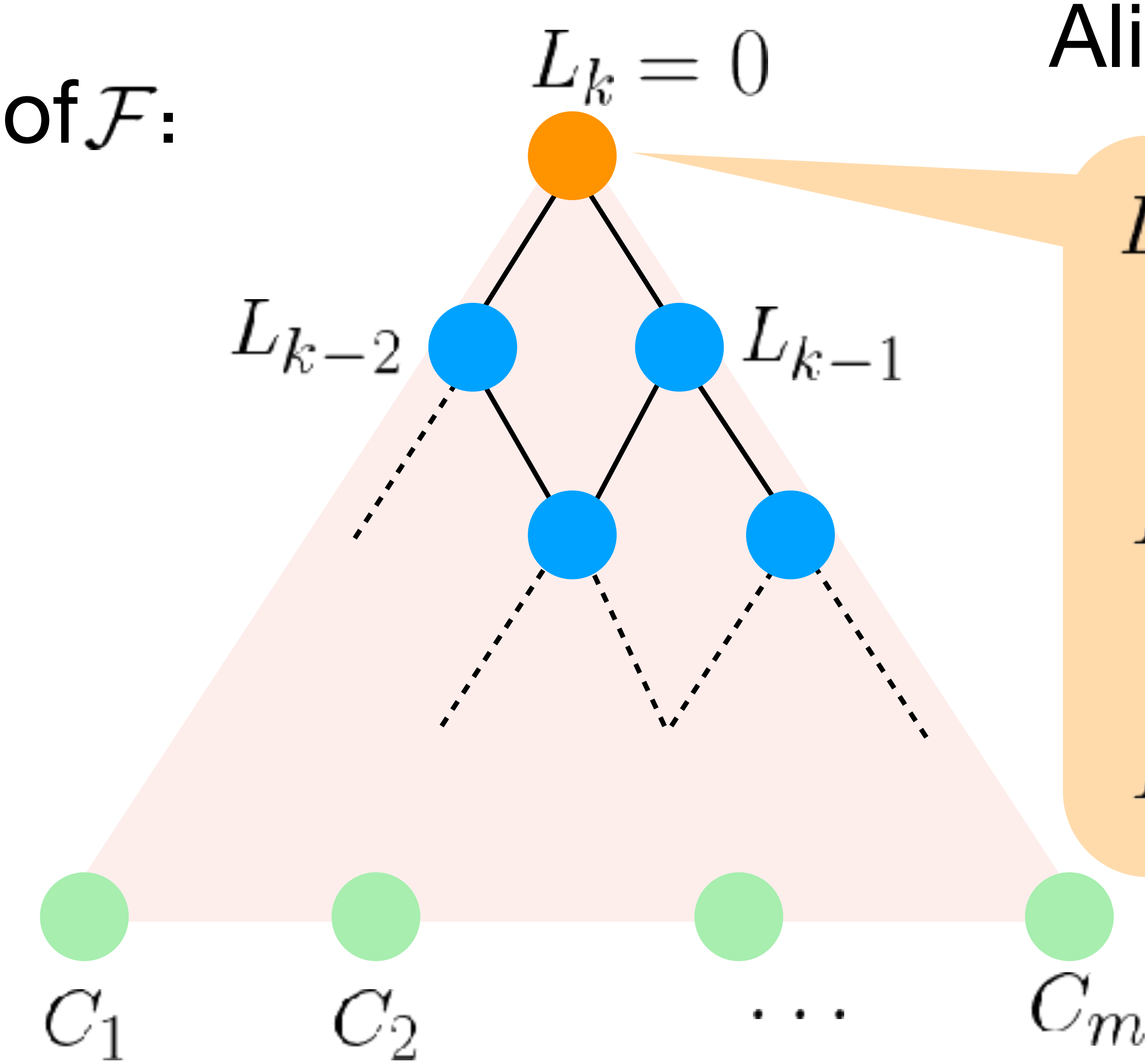

$L_{k-1}, L_{k-2} \models L_k$

Therefore

$L_{k-1}(x,y) = 0$

OR

$L_{k-2}(x,y) = 0$

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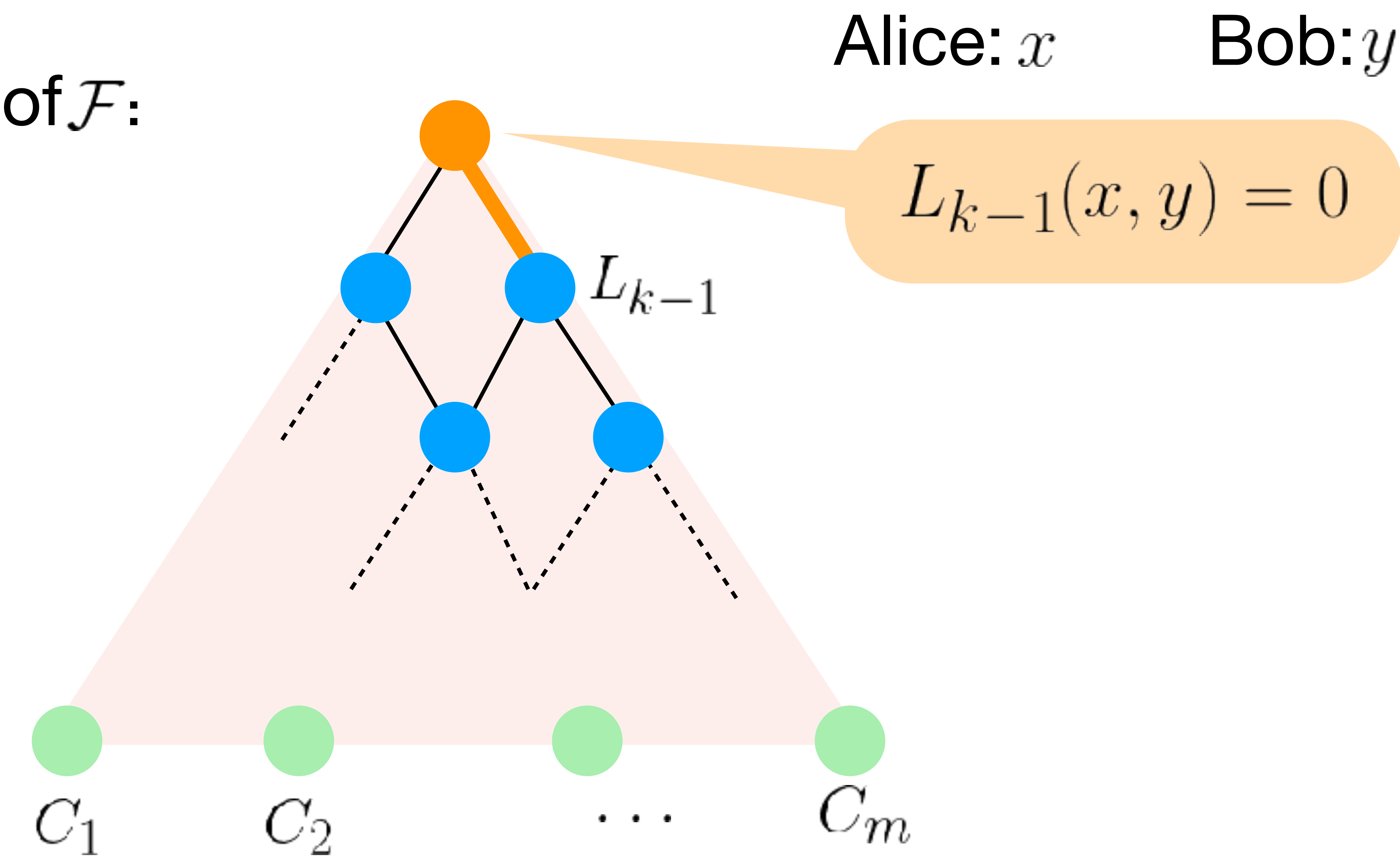

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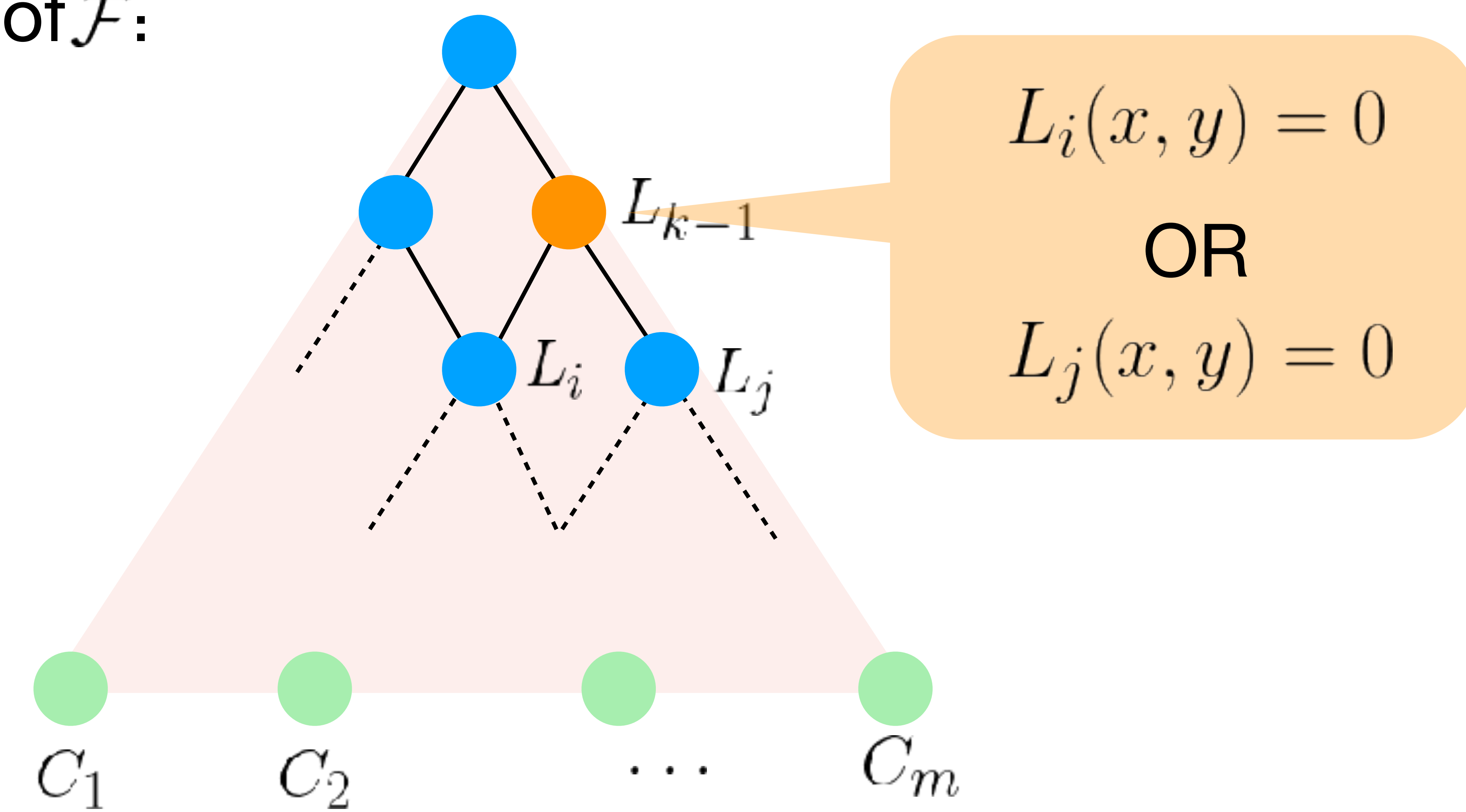

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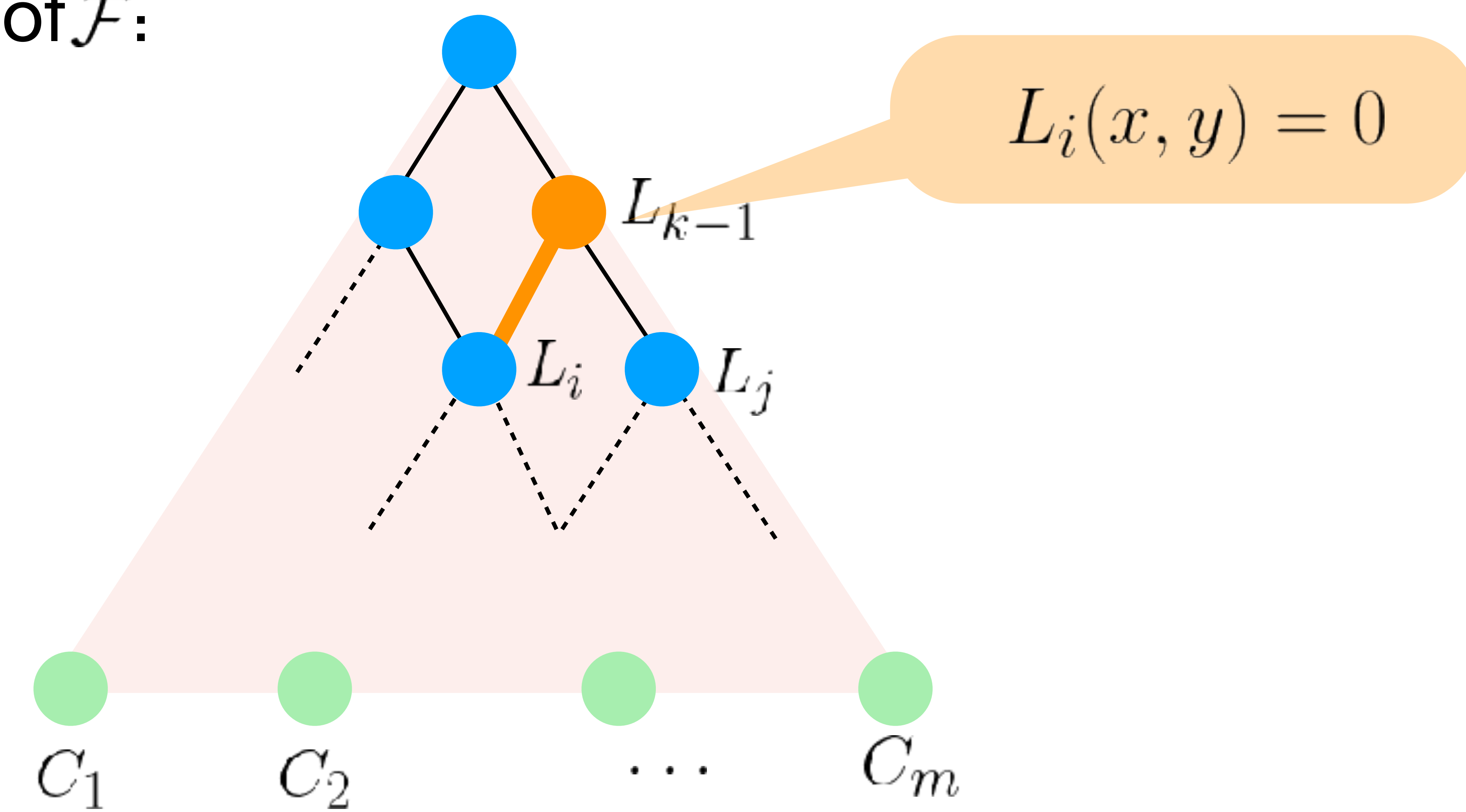

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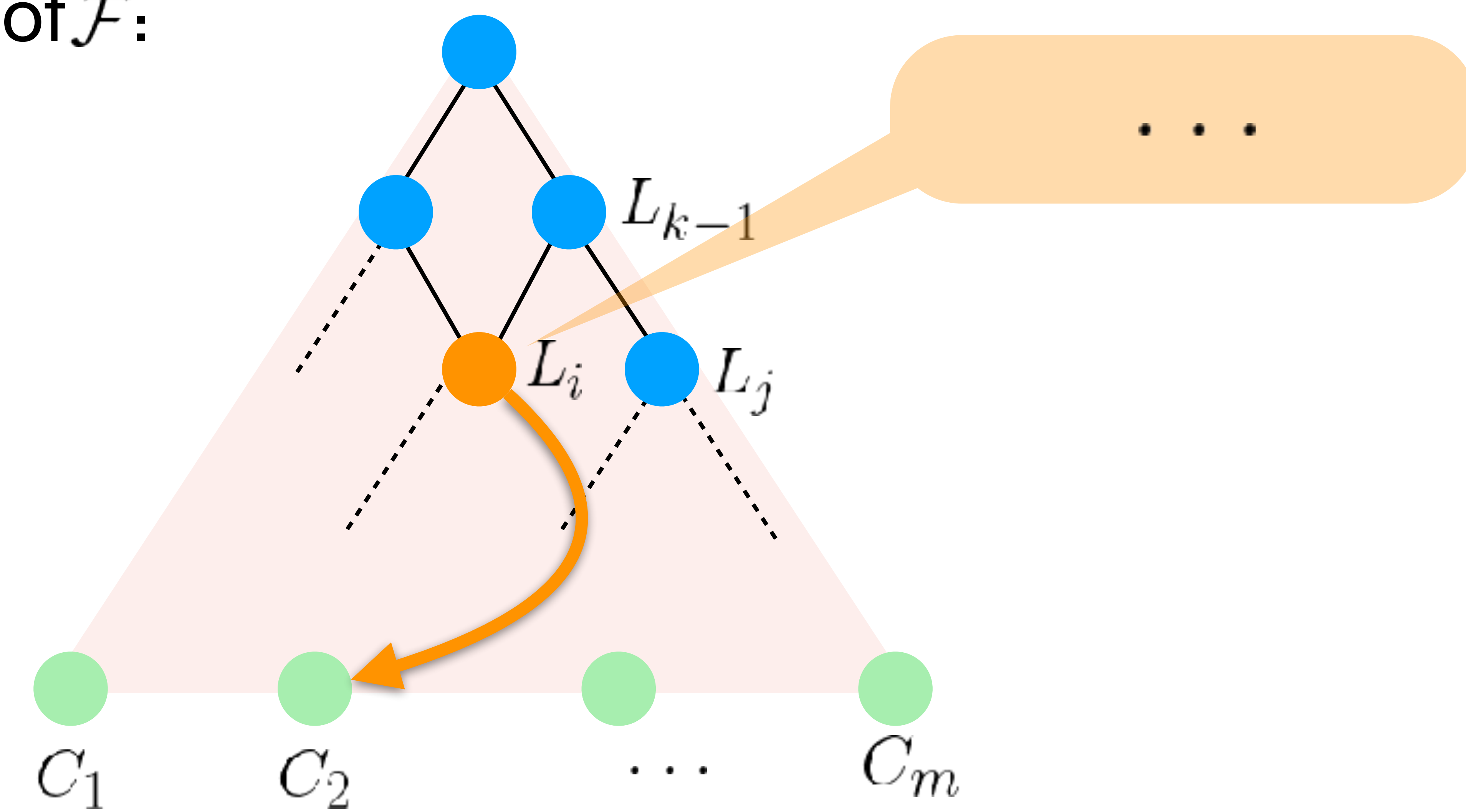

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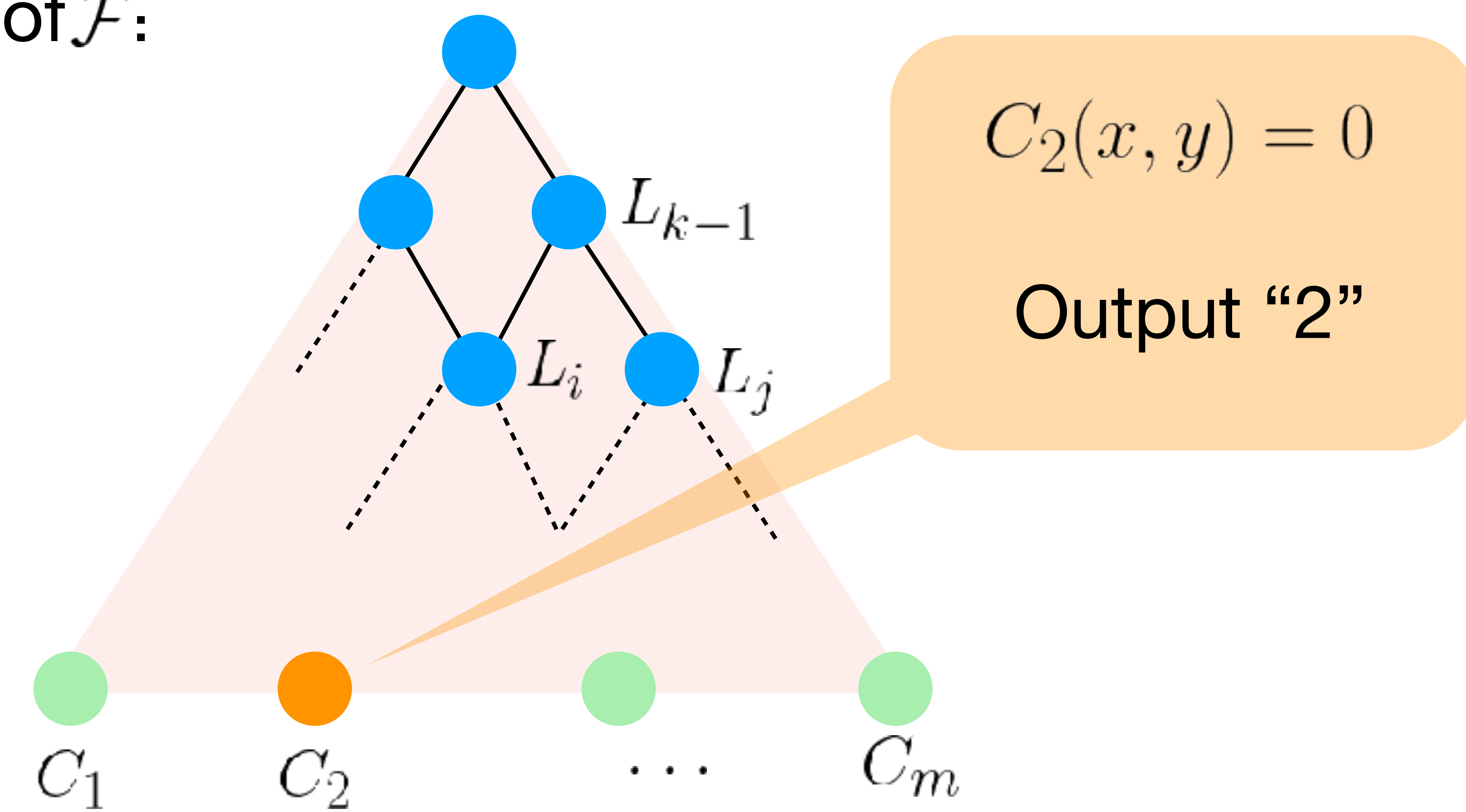

Strategy

CC-Refutation of $\mathcal{F}(X, Y)$

CC-game
for the search problem

CC-game
for the Karchmer-Wigderson game

Monotone circuit
separating minterms from maxterms

Strategy

CC-Refutation of $\mathcal{F}(X, Y)$

CC-game for $\mathrm{Search}_{X,Y}(\mathcal{F})$

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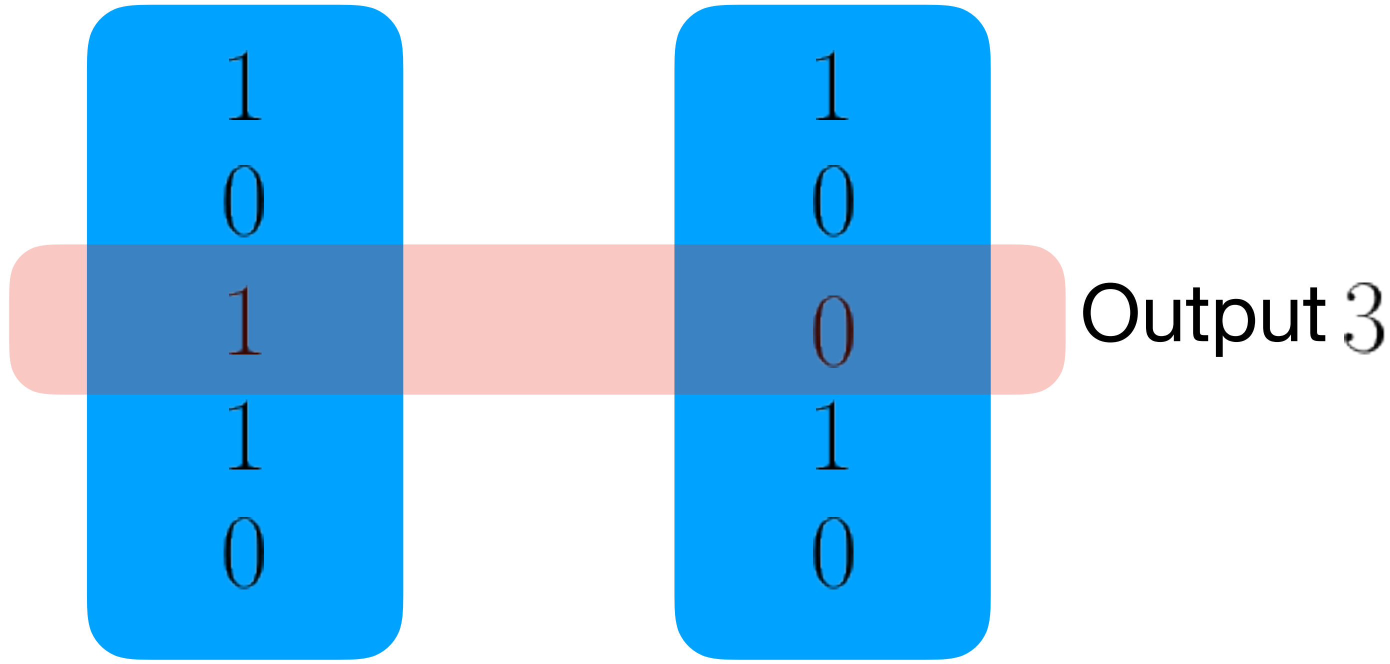
Monotone Karchmer-Wigderson Game

Alice: $x \in \{0,1\}^n$ s.t. $f(x) = 1$ Bob: $y \in \{0,1\}^n$ s.t. $f(y) = 0$



Output: $i \in [n]$ such that $x_i = 1, y_i = 0$

[Razborov 95]: For any partial function monotone function $f : \{0,1\}^n \to \{0,1,*\}$,

monotone CKT size(f) = CC-Game size($KW^+(f)$)

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Strategy

CC-Refutation of $\mathcal{F}(X,Y)$

↓

CC-game for $\mathrm{Search}_{X,Y}(\mathcal{F})$

CC-game for KW^+(minterms, maxterms)

↓

Monotone circuit
separating minterms from maxterms

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CC-Refutation of $\mathcal{F}(X, Y)$

CC-game for $\mathrm{Search}_{X,Y}(\mathcal{F})$

CC-game for KW^+(minterms, maxterms)

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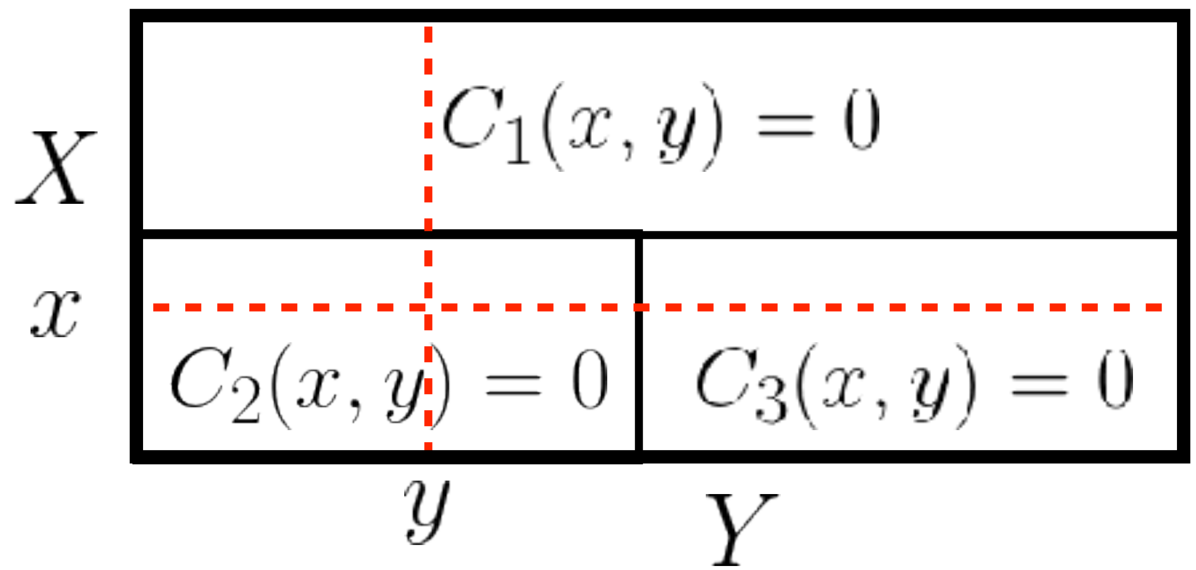
Unsatisfiability Certificate

Unsatisfiable $\mathcal{F}(X,Y) = C_1(x,y) \wedge C_2(x,y) \wedge C_3(x,y)$ over partition $X \cup Y$

Inputs to $\mathrm{Search}_{X,Y}(\mathcal{F})$ Alice: x Bob: y



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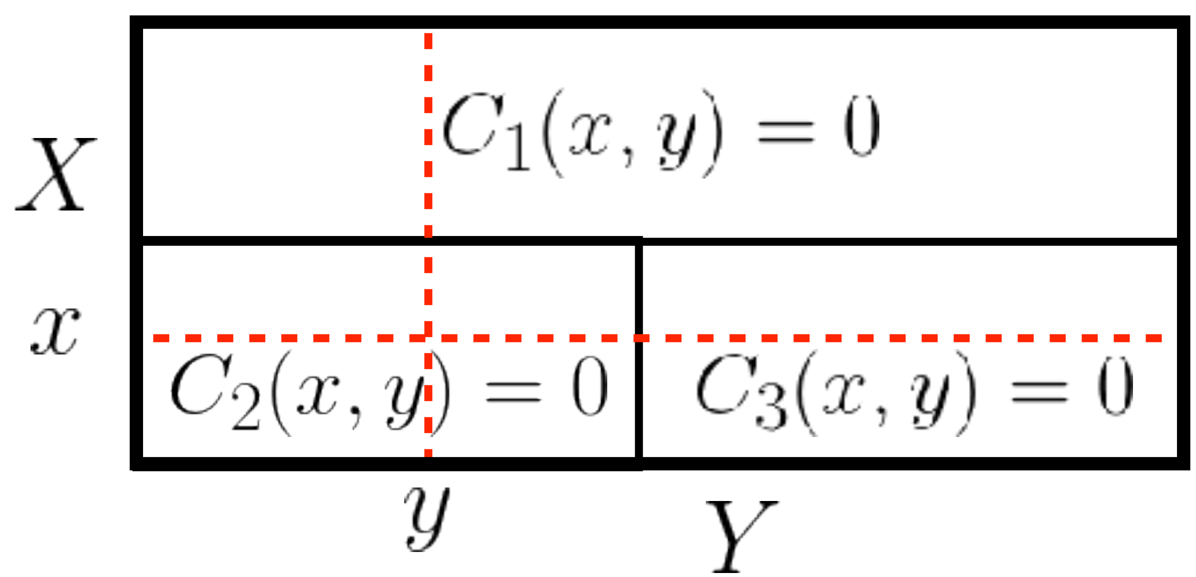


Corresponding inputs to monotone KW game:

		Example	General
Alice:	$x \to \mathcal{U}(x)$	$\mathcal{U}(x) = [0, 1, 1]$	$\mathcal{U}(x)_i = 1 \iff C_i \restriction_X (x) = 0$
Bob:	$y \to \mathcal{V}(y)$	$\mathcal{V}(y) = [0, 0, 1]$	$\mathcal{V}(y)_i = 0 \iff C_i \restriction_Y (y) = 0$

Unsatisfiability Certificate

Alice: $x \to \mathcal{U}(x)$ | $\mathcal{U}(x)_i = 1 \iff C_i \restriction_X (x) = 0$

Bob: $y \to \mathcal{V}(y)$ | $\mathcal{V}(y)_i = 0 \iff C_i \restriction_Y (y) = 0$

Resulting partial function (unsatisfiability certificate):

Minterms: (abuse of notation)

$\mathcal{U}$: The set of outputs of the map $\mathcal{U}$ over all x

Maxterms:

$\mathcal{V}$: The set of outputs of the map $\mathcal{V}$ over all y

Equivalently [HP17],

Unsatisfiability Certificate: $z \in \{0,1\}^m$

$$\mathrm{Certificate}_{\mathcal{F}}(z) = \begin{cases} 1 & \text{if } \{C_i \restriction_X : i \in [m] \setminus z\} \text{ is satisfiable,} \\ 0 & \text{if } \{C_i \restriction_Y : i \in [m]\} \text{ is satisfiable,} \\ * & \text{otherwise} \end{cases}$$

Strategy

CC-Refutation of $\mathcal{F}(X,Y)$

$\Downarrow$

CC-game for $\mathrm{Search}_{X,Y}(\mathcal{F})$

$\Downarrow$

CC-game for KW^+(minterms, maxterms)

$\Downarrow$

Monotone circuit separating $\mathcal{U}$ from $\mathcal{V}$

Theorem: Let $\mathcal{F}$ be an unsatisfiable CNF and $X \cup Y$ be any partition of the variables of $\mathcal{F}$.

CC-Refutation size $\mathcal{F}(X,Y) \approx$ Monotone CKT size ($\mathcal{U}$, $\mathcal{V}$)

Monotone Circuit Lower Bound

$\mathcal{F} \sim \mathcal{F}(m, n, k)$: Choose m clauses of width k uniformly at random with replacement from all possible $\binom{n}{k}2^k$ such clauses

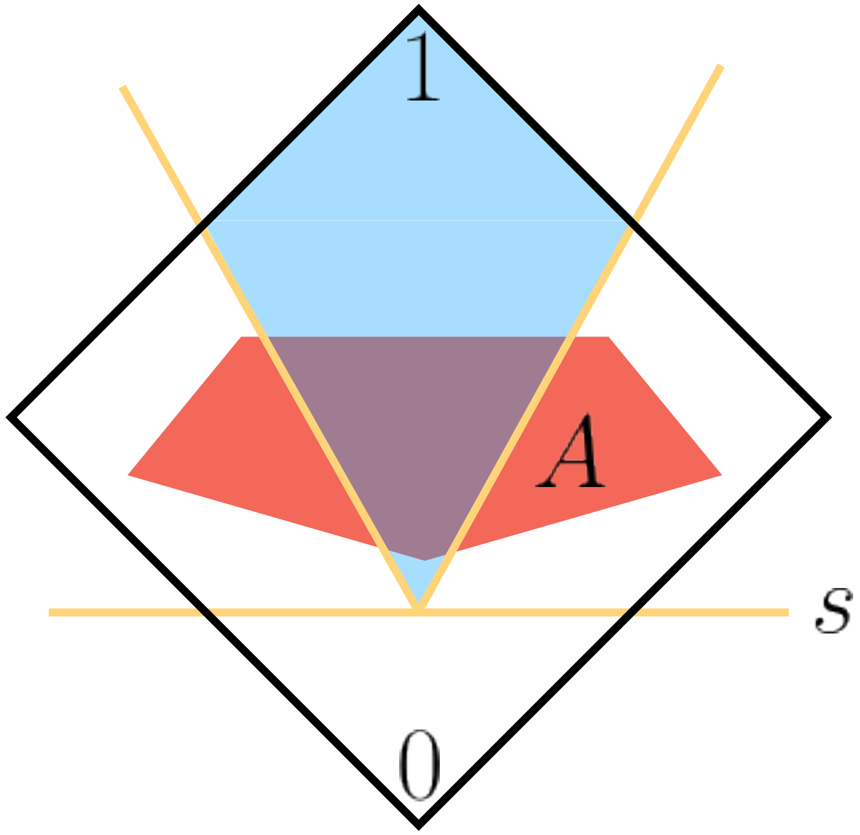
Theorem: Let $m = n^2 2^k$, $k = \theta(\log n)$, and sample $\mathcal{F} \sim \mathcal{F}(m, n, k)$. W.h.p, any monotone circuit separating $\mathcal{U}$ from $\mathcal{V}$ requires $2^{\Omega(n/\log n)}$ gates.

Symmetric Method

Spread out Measure: $A_b(s, A) = \max_{I \subseteq [n]: |I| = s} |\{x \in A : \forall i \in I, x_i = b\}|$

$A_b(s, A)$ small if no set of s variables that, set to b, agrees with a lot of strings in A

$A_1(s, A)$:



Symmetric Method

Spread out Measure: $A_b(s, A) = \max_{I \subseteq [n] : |I| = s} |\{x \in A : \forall i \in I, x_i = b\}|$

Let $f : \{0,1\}^n \to \{0,1\}$ be a partial monotone function

[Jukna99] Any monotone circuit computing f requires at least

$$\min\left\{\frac{|U| - (s-1)A_1(1, U)}{(s-1)^s A_1(s, U)}, \frac{|V|}{(s-1)^s A_0(s, V)}\right\}$$

gates, for any s. Where $U = f^{-1}(1)$, $V = f^{-1}(0)$.

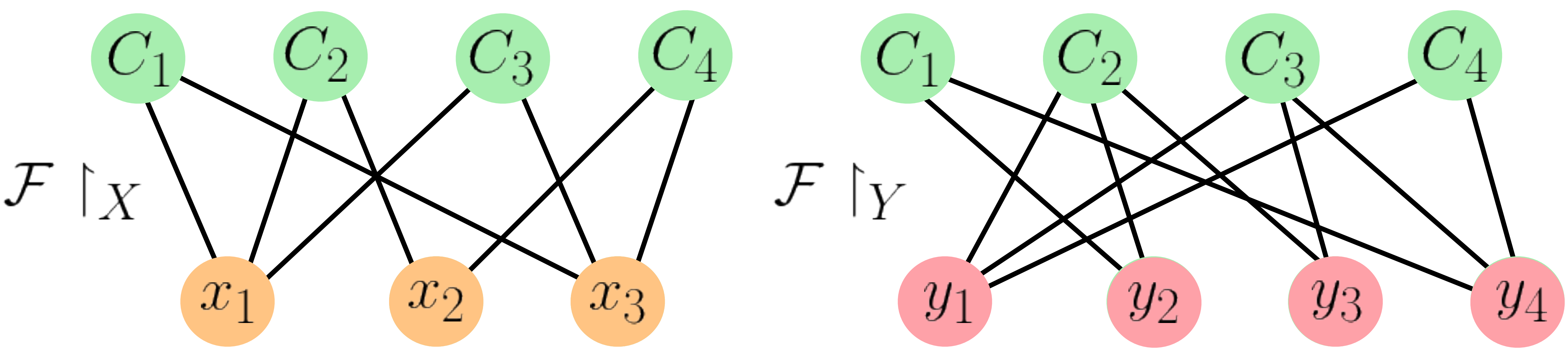
Monotone Circuit Lower Bound

Good expansion properties $\Longrightarrow A_b(s, A)$ is small,

$\mathcal{F} \sim \mathcal{F}(m, n, k)$ is expanding w.h.p.!

Problem

Need $\mathcal{U}$ and $\mathcal{V}$ to be expanding.



That is, we need $\mathcal{F} \restriction_X$ and $\mathcal{F} \restriction_Y$ to be expanding with respect to the fixed variable partition $X \cup Y$.

Balanced Random CNF

Temporary Solution: Change Distribution

$\text{Balanced} - \mathcal{F}(m, n, 2k)$:

1. Sample $\mathcal{F}_X \sim \mathcal{F}(m, n, k)$ on X-variables, $\mathcal{F}_X = C_1(x) \wedge \ldots \wedge C_m(x)$
2. Sample $\mathcal{F}_Y \sim \mathcal{F}(m, n, k)$ on Y-variables, $\mathcal{F}_Y = C_1(y) \wedge \ldots \wedge C_m(y)$

Output: $\mathcal{F}(X, Y) = (C_1(x) \vee C_1(y)) \wedge \ldots \wedge (C_m(x) \vee C_m(y))$

$\mathcal{F} \restriction_X$ and $\mathcal{F} \restriction_Y$ both expanding!

Theorem: Let $m = n^2 2^k$, $k = \theta(\log n)$, and sample $\text{Balanced} - \mathcal{F}(m, n, 2k)$. W.h.p, any monotone circuit separating $\mathcal{U}$ from $\mathcal{V}$ requires $2^{\Omega(n/\log n)}$ gates.

Random CNF

Strategy: Reduce to balanced case!

Sample $\mathcal{F} \sim \mathcal{F}(m, n, k)$, show existence of partition $X \cup Y$, such that

1. Most of the clauses of $\mathcal{F}$ are balanced w.r.t. X and Y,
2. There exists a large set of assignments $\mathcal{A}$ to the X-variables and $\mathcal{B}$ to the Y-variables which satisfy all of the unbalanced clauses.

Apply Symmetric Method of Approximations to $(\mathcal{U}(\mathcal{A}), \mathcal{V}(\mathcal{B}))$

Theorem: Let $m = n^2 2^k$, $k = \theta(\log n)$ and sample $\mathcal{F} \sim \mathcal{F}(m, n, k)$. With high probability, any Cutting Planes refutation of $\mathcal{F}$ requires $2^{\Omega(n/\log n)}$ lines.

Conclusion

- First exponential lower bound on the size of Cutting Planes refutations of random $\theta(\log n)$-CNFs

- Lower bound for random k-CNF for k = constant?
  -Improve symmetric method of approximations, $(s-1)^s$ term in the denominator kills us!

- Cutting Planes lower bound for Tseitin formulas?
  -**Technique incapable of handling Tseitin formulas!**
  - $O(n)$ upper bound on Tseitin in CC.

Thanks!