CSC304 Lecture 18

Fair Division 1: Cake-Cutting

[Image and Illustration (you’ll see!) Credits: Ariel Procaccia]
Cake-Cutting

• A heterogeneous, divisible good
  ➢ Heterogeneous: it may be valued differently by different individuals
  ➢ Divisible: we can share/divide it between individuals

• Represented as $[0,1]$
  ➢ Almost without loss of generality

• Set of players $N = \{1, \ldots, n\}$

• Piece of cake $X \subseteq [0,1]$
  ➢ A finite union of disjoint intervals
Agent Valuations

• Each player $i$ has a valuation $V_i$ that is very much like a probability distribution over $[0,1]$

• Additive: For $X \cap Y = \emptyset$, $V_i(X) + V_i(Y) = V_i(X \cup Y)$

• Normalized: $V_i([0,1]) = 1$

• Divisible: $\forall \lambda \in [0,1]$ and $X$, $\exists Y \subseteq X$ s.t. $V_i(Y) = \lambda V_i(X)$
Fairness Goals

• An allocation is a disjoint partition $A = (A_1, ..., A_n)$ of the cake

• We desire the following fairness properties from our allocation $A$:

• Proportionality (Prop):

$$\forall i \in N: V_i(A_i) \geq \frac{1}{n}$$

• Envy-Freeness (EF):

$$\forall i, j \in N: V_i(A_i) \geq V_i(A_j)$$
Fairness Goals

- **Prop:** $\forall i \in N: V_i(A_i) \geq 1/n$
- **EF:** $\forall i, j \in N: V_i(A_i) \geq V_i(A_j)$

**Question:** What is the relation between proportionality and EF?

1. Prop $\Rightarrow$ EF
2. EF $\Rightarrow$ Prop
3. Equivalent
4. Incomparable
**Cut-and-Choose**

- Algorithm for $n = 2$ players

- Player 1 divides the cake into two pieces $X, Y$ s.t.
  \[ V_1(X) = V_1(Y) = 1/2 \]

- Player 2 chooses the piece she prefers.

- This is EF and therefore proportional.

  ➢ Why?
Input Model

• How do we measure the “time complexity” of a cake-cutting algorithm for $n$ players?

• Typically, time complexity is a function of the length of input encoded as binary.

• Our input consists of functions $V_i$, which requires infinite bits to encode.

• We want running time just as a function of $n$. 
Robertson-Webb Model

• We restrict access to valuations $V_i$’s through two types of queries:
  ➢ $Eval_i(x, y)$ returns $V_i([x, y])$
  ➢ $Cut_i(x, \alpha)$ returns $y$ such that $V_i([x, y]) = \alpha$
Robertson-Webb Model

• Two types of queries:
  ➢ $\text{Eval}_i(x, y) = V_i([x, y])$
  ➢ $\text{Cut}_i(x, \alpha) = y \text{ s.t. } V_i([x, y]) = \alpha$

• **Question:** How many queries are needed to find an EF allocation when $n = 2$?

• **Answer:** 2
  ➢ Why?
Dubins-Spanier

• Protocol for finding a proportional allocation for \( n \) players

  • Referee starts at 0, and continuously moves knife to the right.
  • Repeat: when piece to the left of knife is worth \( \frac{1}{n} \) to a player, the player shouts “stop”, gets the piece, and exits.
  • The last player gets the remaining piece.
Moving knife is not really needed.

At each stage, we can ask each remaining player a cut query to mark his $1/n$ point in the remaining cake.

Move the knife to the leftmost mark.
DUBINS-SPANIER
DUBINS-SPANIER
Dubins-Spanier
Dubins-Spanier

• Question: What is the complexity of the Dubins-Spanier protocol in the Robertson-Webb model?

1. $\Theta(n)$
2. $\Theta(n \log n)$
3. $\Theta(n^2)$
4. $\Theta(n^2 \log n)$
EVEN-PAZ

• Input: Interval \([x, y]\), number of players \(n\)
  ➢ Assume \(n = 2^k\) for some \(k\)
• If \(n = 1\), give \([x, y]\) to the single player.
• Otherwise, let each player \(i\) mark \(z_i\) s.t.
  \[
  V_i([x, z_i]) = \frac{1}{2} V_i([x, y])
  \]
• Let \(z^*\) be the \(n/2\) mark from the left.
• Recurse on \([x, z^*]\) with the left \(n/2\) players, and on \([z^*, y]\) with the right \(n/2\) players.
**Theorem:** \text{EVEN-PAZ} returns a Prop allocation.

**Proof:**

- Inductive proof. We want to prove that if player \(i\) is allocated piece \(A_i\) when \([x, y]\) is divided between \(n\) players, \(V_i(A_i) \geq (1/n) V_i([x, y])\)
  - Then Prop follows because initially \(V_i([x, y]) = V_i([0,1]) = 1\)
- Base case: \(n = 1\) is trivial.
- Suppose it holds for \(n = 2^{k-1}\). We prove for \(n = 2^k\).
- Take the \(2^{k-1}\) left players.
  - Every left player \(i\) has \(V_i([x, z^*]) \geq (1/2) V_i([x, y])\)
  - If it gets \(A_i\), by induction, \(V_i(A_i) \geq \frac{1}{2^{k-1}} V_i([x, z^*]) \geq \frac{1}{2^k} V_i([x, y])\)
Even-Paz

\[ T(1) = 0, T(n) = 2n + 2T\left(\frac{n}{2}\right) \]

Overall: \(2n \log n\)
Complexity of Proportionality

- Theorem [Edmonds and Pruhs, 2006]: Any proportional protocol needs $\Omega(n \log n)$ operations in the Robertson-Webb model.

- Thus, the EVEN-PAZ protocol is (asymptotically) provably optimal!
Envy-Freeness?

• “I suppose you are also going to give such cute algorithms for finding envy-free allocations?”

• Bad luck. For $n$-player EF cake-cutting:
  - [Procaccia 2009] shows $\Omega(n^2)$ lower bound for EF.
  - Last year, the long-standing major open question of “bounded EF protocol” was resolved!

  - [Aziz and Mackenzie, 2016]: $O(n^{nnnn})$ protocol!
    - Yes, it’s not a typo. Go figure!
Next Lecture

- Strategyproofness
- Pareto optimality
- Restricted case of multiple homogeneous goods
- Generalization to the case of indivisible goods