CSC304 Lecture 16

Voting 3: Axiomatic, Statistical, and Utilitarian Approaches to Voting
Announcements

• Assignment 2 was due today at 3pm

• If you have grace credits left (check MarkUs), you could take up to two more days, and submit by Wed 3pm

• On Wednesday, we will go over solutions to A2 problems in class
  ➢ We’ll do a Piazza poll to find the most popular questions, and solve them first
Recap

• We introduced a plethora of voting rules
  ➢ Plurality
  ➢ Borda
  ➢ Veto
  ➢ \( k \)-Approval
  ➢ STV
  ➢ Plurality with runoff
  ➢ Kemeny
  ➢ Copeland
  ➢ Maximin

• Which is the right way to aggregate preferences?
  ➢ GS Theorem: There is no good strategyproof voting rule.
  ➢ For now, let us forget about incentives. Let us focus on how to aggregate given truthful votes.
Recap

- Set of voters $N = \{1, \ldots, n\}$
- Set of alternatives $A$, $|A| = m$
- Voter $i$ has a preference ranking $\succ_i$ over the alternatives

- Preference profile $\succ = \text{collection of all voter rankings}$
- Voting rule (social choice function) $f$
  - Takes as input a preference profile $\succ$
  - Returns an alternative $a \in A$
Axiomatic Approach

• An axiom is a desideratum in which we require a voting rule to behave in a specific way.

• **Goal:** define a set of reasonable axioms, and search for voting rules that satisfy them
  ➢ **Ultimate hope:** find that a unique voting rule satisfies the axioms we are interested in!

• Sadly, we often find the opposite.
  ➢ Many combinations of reasonable axioms cannot be satisfied by any voting rule.
  ➢ E.g., the GS theorem (nondictatorship, ontoness, strategyproofness), Arrow’s theorem (will see), ...
Axiomatic Approach

• Weak axioms, satisfied by all popular voting rules

• **Unanimity**: If all voters have the same top choice, that alternative is the winner.

\[
\text{top}(>_i) = a \forall i \in N \Rightarrow f(\succ) = a
\]

➢ An even weaker version requires all rankings to be identical

• **Pareto optimality**: If all voters prefer \( a \) to \( b \), then \( b \) is not the winner.

\[
(a >_i b \forall i \in N) \Rightarrow f(\succ) \neq b
\]

• Q: *What is the relation between these axioms?*

➢ **Pareto optimality** ⇒ **Unanimity**
Axiomatic Approach

• **Anonymity:** Permuting votes does not change the winner (i.e., voter identities don’t matter).
  - E.g., these two profiles must have the same winner:
    - \{voter 1: \(a > b > c\), voter 2: \(b > c > a\)\}
    - \{voter 1: \(b > c > a\), voter 2: \(a > b > c\)\}

• **Neutrality:** Permuting alternative names just permutes the winner.
  - E.g., say \(a\) wins on \{voter 1: \(a > b > c\), voter 2: \(b > c > a\)\}
  - We permute all names: \(a \rightarrow b\), \(b \rightarrow c\), and \(c \rightarrow a\)
  - New profile: \{voter 1: \(b > c > a\), voter 2: \(c > a > b\)\}
  - Then, the new winner must be \(b\).
Axiomatic Approach

• Neutrality is tricky
  ➢ As we have it now, it is inconsistent with anonymity!
    o Imagine {voter 1: \(a > b\), voter 2: \(b > a\)}
    o Without loss of generality, say \(a\) wins
    o Imagine a different profile: {voter 1: \(b > a\), voter 2: \(a > b\)}
      • Neutrality: We just exchanged \(a \leftrightarrow b\), so winner is \(b\).
      • Anonymity: We just exchanged the votes, so winner stays \(a\).

➢ Typically, we only require neutrality for...
  o Randomized rules: E.g., a rule could satisfy both by choosing \(a\) and \(b\) as the winner with probability \(\frac{1}{2}\) each, on both profiles
  o Deterministic rules that return a set of tied winners: E.g., a rule could return \(\{a, b\}\) as tied winners on both profiles.
Axiomatic Approach

• Stronger but more subjective axioms

• **Majority consistency:** If a majority of voters have the same top choice, that alternative wins.

\[
\left( |\{i: \text{top}(>_i) = a\}| > \frac{n}{2} \right) \Rightarrow f(\vec{>}) = a
\]

• **Condorcet consistency:** If \(a\) defeats every other alternative in a pairwise election, \(a\) wins.

\[
\left( |\{i: a >_i b\}| > \frac{n}{2}, \forall b \neq a \right) \Rightarrow f(\vec{>}) = a
\]

• **Q:** What is the relation between these two?

  ➢ *Condorcet consistency* \(\Rightarrow\) *Majority consistency*
Axiomatic Approach

• **Majority consistency:** If a majority of voters have the same top choice, that alternative wins.

• **Condorcet consistency:** If $a$ defeats every other alternative in a pairwise election, $a$ wins.

• **Question:** Which of these does *plurality* satisfy?
  - A. Both.
  - B. Only majority consistency.
  - C. Only Condorcet consistency.
  - D. Neither.
Axiomatic Approach

• **Majority consistency:** If a majority of voters have the same top choice, that alternative wins.

• **Condorcet consistency:** If $a$ defeats every other alternative in a pairwise election, $a$ wins.

• **Question:** Which of these does *Borda count* satisfy?
  - A. Both.
  - B. Only majority consistency.
  - C. Only Condorcet consistency.
  - D. Neither.
Axiomatic Approach

• **Majority consistency**: If a majority of voters have the same top choice, that alternative wins.

• **Condorcet consistency**: If $a$ defeats every other alternative in a pairwise election, $a$ wins.

• **Fun fact about Condorcet consistency**
  - Most rules that “focus on positions” (positional scoring rules, STV, plurality with runoff) violate it
  - Most rules that “focus on pairwise comparisons” (Kemeny, Copeland, Maximin) satisfy it
Axiomatic Approach

• Is even the weaker axiom majority consistency a reasonable one to expect?

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Piazza Poll: Do you think we should require that the voting rule must output \( a \) irrespective of how tall the gray region is?
Axiomatic Approach

• **Consistency:** If \( a \) is the winner on two profiles, it must be the winner on their union.

\[
f(\succ_1) = a \land f(\succ_2) = a \Rightarrow f(\succ_1 + \succ_2) = a
\]

- Example: \( \succ_1 = \{a > b > c\} \), \( \succ_2 = \{a > c > b, b > c > a\} \)
- Then, \( \succ_1 + \succ_2 = \{a > b > c, a > c > b, b > c > a\} \)

• Do you think consistency must be satisfied?
  - Young [1975] showed that subject to mild requirements, a voting rule is consistent if and only if it is a positional scoring rule!
  - Thus, plurality with runoff, STV, Kemeny, Copeland, Maximin, etc are *not* consistent.
Axiomatic Approach

• **Weak monotonicity:** If \( a \) is the winner, and \( a \) is “pushed up” in some votes, \( a \) remains the winner.

  \[ f(\succ) = a \Rightarrow f(\succ') = a \text{ if} \]
  1. \( b \succ_i c \iff b \succ'_i c, \forall i \in N, b, c \in A\{a\} \]
     “Order among other alternatives preserved in all votes”
  2. \( a \succ_i b \Rightarrow a \succ'_i b, \forall i \in N, b \in A\{a\} \) (\( a \) only improves)
     “In every vote, \( a \) still defeats all the alternatives it defeated”

• Contrast: strong monotonicity requires \( f(\succ') = a \) even if \( \succ' \) only satisfies the 2\textsuperscript{nd} condition

  \[ \text{It is thus too strong. Equivalent to strategyproofness!} \]
  \[ \text{Only satisfied by dictatorial/non-onto rules [GS theorem]} \]
Axiomatic Approach

• **Weak monotonicity:** If $a$ is the winner, and $a$ is “pushed up” in some votes, $a$ remains the winner.

  $f(\succ) = a \Rightarrow f(\succ') = a$, where
  
  $b >_i c \iff b >'_i c, \forall i \in N, b, c \in A\{a}$ (Order of others preserved)
  
  $a >_i b \Rightarrow a >'_i b, \forall i \in N, b \in A\{a}$ (a only improves)

• **Weak monotonicity is satisfied by most voting rules**

  Only exceptions (among rules we saw):
  STV and plurality with runoff

  But this helps STV be hard to manipulate

  [Conitzer & Sandholm 2006]: “Every weakly monotonic voting rule is easy to manipulate on average.”
Axiomatic Approach

- STV violates weak monotonicity

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<th>5 voters</th>
<th>2 voters</th>
<th>6 voters</th>
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<tr>
<td>a</td>
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- First $c$, then $b$ eliminated
- Winner: $a$

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- First $b$, then $a$ eliminated
- Winner: $c$
Good news: The material in the slides that follow is *not* part of the syllabus.

- It is to give you a flavor of other interesting results/approaches in voting.

Bad news: That’s because I’m going to go over it really fast!
Axiomatic Approach

• Arrow’s Impossibility Theorem
  ➢ Applies to social welfare functions (want a consensus ranking)
  ➢ Independence of Irrelevant Alternatives (IIA): If the preferences of all voters between $a$ and $b$ are unchanged, the social preference between $a$ and $b$ should not change
    o Criticized to be too strong
  ➢ Theorem: IIA cannot be achieved together with Pareto optimality (if all prefer $a$ to $b$, social preference should be $a \succ b$) unless the rule is a dictatorship.
  ➢ Arrow’s theorem set the foundations for the axiomatic approach to voting
Statistical Approach

• Assume that there is a ground truth ranking $\sigma^*$

• Votes $\{\succ_i\}$ are generated i.i.d. from a distribution parametrized by $\sigma^*$
  ➢ Formally, there is a probability distribution $Pr[\cdot | \sigma]$ for every ranking $\sigma$
  ➢ $Pr[\succ | \sigma]$ denotes the probability of drawing a vote $\succ$ given that the ground truth is $\sigma$

• Use **maximum likelihood estimate (MLE)** of the ground truth
  ➢ Given $\vec{\succ}$, return $\arg\max_{\sigma} (Pr[\vec{\succ} | \sigma] = \prod_{i=1}^{n} Pr[\succ_i | \sigma])$
Statistical Approach

• Example: Mallows’ model

➢ Recall: Kendall-tau distance $d$ between two rankings is the #pairs of alternatives whose comparisons they differ on

➢ Mallows’ model: $\Pr[\sigma] \propto \varphi^{d(\sigma, \sigma)}$, where
  o $\varphi \in (0,1]$ is the “noise parameter”
  o $\varphi \rightarrow 0$ means the distribution becomes accurate ($\Pr[\sigma | \sigma] \rightarrow 1$)
  o $\varphi = 1$ represents the uniform distribution
  o Normalization constant $Z_\varphi = \sum_\sigma \varphi^{d(\sigma, \sigma)}$ does not depend on $\sigma$

➢ The greater the distance from the ground truth, the smaller the probability
Statistical Approach

• Example: Mallows’ model

➢ What is the MLE ranking for Mallows’ model?

\[
\max_{\sigma} \prod_{i=1}^{n} \Pr[\succ_i | \sigma] = \max_{\sigma} \prod_{i=1}^{n} \frac{\varphi^{d(\succ_i, \sigma)}}{Z_\varphi} = \max_{\sigma} \frac{\varphi^{\sum_{i=1}^{n} d(\succ_i, \sigma)}}{Z_\varphi}
\]

➢ The MLE ranking minimizes \(\sum_{i=1}^{n} d(\succ_i, \sigma)\)

➢ This is precisely the Kemeny ranking!

• Statistical approach yields a unique rule, but is specific to the assumed distribution of votes
Utilitarian Approach

• Assume that voters have numerical utilities \( \{v_i(a)\} \)
• Their votes reflect comparisons of utilities:
  \[ a >_i b \iff v_i(a) \geq v_i(b) \]
• Goal:
  ➢ Select \( a^* \) with the maximum social welfare \( \sum_i v_i(a^*) \)
  ➢ Cannot achieve this if we just know comparisons of utilities
    o Select \( a^* \) that gives the best worst-case approximation of welfare
      (ratio of maximum social welfare to social welfare achieved)

\[
\min_a \max_{\{v_i \text{ consistent with } >_i\}} \frac{\max_b \sum_i v_i(b)}{\sum_i v_i(a)}
\]
Utilitarian Approach

• **Pros:** Uses minimal subjective assumptions and yet yields a unique voting rule

• **Cons:** Difficult to compute and unintuitive to humans

• This approach is currently deployed on RoboVote
  - It has been extended to select a set of alternatives
  - My ongoing work: use it to select a consensus ranking
    - Results in a large, nonconvex, quadratically constrained quadratic program