CSC304 Lecture 15

Voting 2: Gibbard-Satterthwaite Theorem
Recap

• We introduced a plethora of voting rules
  - Plurality
  - Borda
  - Veto
  - $k$-Approval
  - STV
  - Plurality with runoff
  - Kemeny
  - Copeland
  - Maximin

• All these rules do something reasonable on a given preference profile
  - Only makes sense if preferences are truthfully reported
Recap

• Set of voters $N = \{1, \ldots, n\}$
• Set of alternatives $A$, $|A| = m$
• Voter $i$ has a preference ranking $\succ_i$ over the alternatives

• Preference profile $\succ = \text{collection of all voter rankings}$
• Voting rule (social choice function) $f$
  - Takes as input a preference profile $\succ$
  - Returns an alternative $a \in A$

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\begin{array}{|c|c|c|}
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1 & 2 & 3 \\
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a & c & b \\
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b & a & a \\
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c & b & c \\
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\end{array}
\]
Strategyproofness

• Would any of these rules incentivize voters to report their preferences truthfully?

• A voting rule $f$ is strategyproof if for every
  - preference profiles $\succ$,  
  - voter $i$, and  
  - preference profile $\succ'$ such that $\succ'_j = \succ_j$ for all $j \neq i$

  it is not the case that $f(\succ') >_i f(\succ)$
Strategyproofness

• None of the rules we saw are strategyproof!

• Example: Borda Count
  - In the true profile, $b$ wins
  - Voter 3 can make $a$ win by pushing $b$ to the end

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Winner: $b$

Winner: $a$
Borda’s Response to Critics

My scheme is intended only for honest men!

Random 18th century French dude
Strategyproofness

• Are there any strategyproof rules?
  ➢ Sure

• Dictatorial voting rule
  ➢ The winner is always the most preferred alternative of voter $i$

• Constant voting rule
  ➢ The winner is always the same

• Not satisfactory (for most cases)
Three Properties

• **Strategyproof**: Already defined. No voter has an incentive to misreport.

• **Onto**: Every alternative can win under some preference profile.

• **Nondictatorial**: There is no voter $i$ such that $f(\rightarrow)$ is always the top alternative for voter $i$. 
Gibbard-Satterthwaite

• **Theorem:** For $m \geq 3$, no deterministic social choice function can be strategyproof, onto, and nondictatorial simultaneously 😞

• **Proof:** We will prove this for $n = 2$ voters.

  ➢ Step 1: Show that SP is equivalent to “strong monotonicity” [HW 3?]

  ➢ **Strong Monotonicity (SM):** If $f(\succ) = a$, and $\succ'$ is such that $\forall i \in N, x \in A: a \succ_i x \Rightarrow a \succ'_i x$, then $f(\succ') = a$.
    o If $a$ still defeats every alternative it defeated in every vote in $\succ$, it should still win.
Gibbard-Satterthwaite

• **Theorem**: For $m \geq 3$, no deterministic social choice function can be strategyproof, onto, and nondictatorial simultaneously 😞

• **Proof**: We will prove this for $n = 2$ voters.

  ➢ Step 2: Show that SP+onto implies “Pareto optimality” [HW 3?]

  ➢ **Pareto Optimality (PO)**: If $a \succ_i b$ for all $i \in N$, then $f(\succ) \neq b$.
    - If there is a different alternative that everyone prefers, your choice is not Pareto optimal (PO).
Gibbard-Satterthwaite

- **Proof for n=2:** Consider problem instance $I(a, b)$

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Say $f(\succ_1, \succ_2) = a$

- PO: $f(\succ_1, \succ_2) \in \{a, b\}$

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- PO: $f(\succ_1, \succ'_2) \in \{a, b\}$
- SP: $f(\succ_1, \succ'_2) \neq b$

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- Due to strong monotonicity

$I(a, b)$
Gibbard-Satterthwaite

• Proof for n=2:
  ➢ If \( f \) outputs \( a \) on instance \( I(a, b) \), voter 1 can get \( a \) elected whenever she puts \( a \) first.
    o In other words, voter 1 becomes dictatorial for \( a \).
    o Denote this by \( D(1, a) \).
  ➢ If \( f \) outputs \( b \) on \( I(a, b) \)
    o Voter 2 becomes dictatorial for \( b \), i.e., we have \( D(2, b) \).

• For every \( I(a, b) \), we have \( D(1, a) \) or \( D(2, b) \).
Gibbard-Satterthwaite

• Proof for $n=2$:
  ➢ On some $I(a^*, b^*)$, suppose $D(1, a^*)$ holds.
  ➢ Then, we show that voter 1 is a dictator. That is, $D(1, b)$ must hold for every other $b$ as well.
  ➢ Take $b \neq a$. Because $|A| \geq 3$, there exists $c \in A \setminus \{a^*, b\}$.
  ➢ Consider $I(b, c)$. We either have $D(1, b)$ or $D(2, c)$.
  ➢ But $D(2, c)$ is incompatible with $D(1, a^*)$
    o Who would win if voter 1 puts $a^*$ first and voter 2 puts $c$ first?
  ➢ Thus, we have $D(1, b)$, as required.
  ➢ QED!
Circumventing G-S

• Randomization
  ➢ Gibbard characterized all randomized strategyproof rules
  ➢ Somewhat better, but still too restrictive

• Restricted preferences
  ➢ Median for facility location (more generally, for single-peaked preferences on a line)
  ➢ Will see other such settings later

• Money
  ➢ E.g., VCG is nondictatorial, onto, and strategyproof, but charges payments to agents
Circumventing G-S

• Equilibrium analysis
  ➢ Maybe good alternatives still win under Nash equilibria?

• Lack of information
  ➢ Maybe voters don’t know how other voters will vote?
Circumventing G-S

• Computational complexity (Bartholdi et al.)
  ➢ Maybe the rule is manipulable, but it is NP-hard to find a successful manipulation?
  ➢ Groundbreaking idea! NP-hardness can be good!!

• Not NP-hard: plurality, Borda, veto, Copeland, maximin, ...

• NP-hard: Copeland with a peculiar tie-breaking, STV, ranked pairs, ...
Circumventing G-S

• Computational complexity
  ➢ Unfortunately, NP-hardness just says it is hard for some worst-case instances.
  ➢ What if it is actually easy for most practical instances?
  ➢ Many rules admit polynomial time manipulation algorithms for fixed #alternatives ($m$)
  ➢ Many rules admit polynomial time algorithms that find a successful manipulation on almost all profiles (the fraction of profiles converges to 1)

• Interesting open problem to design voting rules that are hard to manipulate on average
Social Choice

• Let’s forget incentives for now.
• Even if voters reveal their preferences truthfully, we do not have a “right” way to choose the winner.

• Who is the right winner?
  ➢ On profiles where the prominent voting rules have different outputs, all answers seem reasonable [HW3].
Axiomatic Approach

• Define axiomatic properties we may want from a voting rule

• We already defined some:
  ➢ Majority consistent
  ➢ Condorcet consistent
  ➢ Onto
  ➢ Strategyproof
  ➢ Strongly monotonic
  ➢ Pareto optimal
Axiomatic Approach

• We will see four more:
  ➢ Unanimity
  ➢ Weak monotonicity
  ➢ Consistency (!)
  ➢ Independence of irrelevant alternatives (IIA)

• Problem?
  ➢ Cannot satisfy many interesting combinations of properties
  ➢ Arrow’s impossibility result
  ➢ Other similar impossibility results
Other Approaches?

• Statistical
  ➢ There exists an objectively true answer
    o E.g., say the question is: “Sort the candidates by the #votes they will receive in an upcoming election.”
  ➢ Every voter is trying to estimate the true ranking
  ➢ Goal is to find the most likely ground truth given votes

• Utilitarian
  ➢ Back to “numerical” welfare maximization, but we still ask voters to only report ranked preferences
    o $a \succ_i b \succ_i c$ simply means $v_i(a) \geq v_i(b) \geq v_i(c)$
  ➢ How well can we maximize welfare subject to such partial information?