CSC304 Lecture 12

Ending Mechanism Design w/ Money:
Recap revenue maximization
& Myerson’s auction

Begin Mechanism Design w/o Money:
Facility Location
Recap

- Single-item auction with 1 seller, \( n \) buyers
- Buyer \( i \) has value \( v_i \) drawn from cdf \( F_i \) (pdf \( f_i \))
- Virtual value function: \( \varphi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \)
- Myerson’s theorem: \( E[\text{Revenue}] = E[\sum_i \varphi_i(v_i) * x_i] \)
  - Maximize revenue = maximize virtual welfare subject to monotonic allocation rule
Recap

• When all $F_i$’s are regular
  ➢ Monotonicity is automatic

• Allocation: Give to agent $i$ with maximum $\varphi_i(v_i)$ if $\varphi_i(v_i) \geq 0$
  ➢ When the maximum $\varphi_i(v_i)$ is negative, not selling the item is better (zero virtual welfare > negative virtual welfare)

• Payment: Charge
  $$v_i^* = \min\{v_i' : \varphi_i(v_i') \geq \max(0, \varphi_j(v_j)) \forall j \neq i\}$$
  ➢ Least possible value for which the agent still gets the item
  ➢ If virtual value drops below any other virtual value or below 0, the agent loses the item
Recap

• Special case: All $F_i = F = \text{Regular}$
  ➢ VCG with reserve price $\varphi^{-1}(0)$

• Allocation: Give the item to agent $i$ with the maximum value $v_i$ but only if $v_i \geq \varphi^{-1}(0)$
  ➢ Equivalent to $\varphi(v_i) \geq 0$

• Payment: $\max \left( \varphi^{-1}(0), \max_{j \neq i} v_j \right)$
  ➢ Least possible value for which the agent still gets the item
  ➢ The agent loses the item as soon as his value goes below either the 2$^{nd}$ highest bid or the reserve price
Approx. Optimal Auctions

• When $F_i$’s are complex, the virtual valuation function is complex too
  ➢ The optimal auction is unintuitive
  ➢ Two simple auctions that achieve good revenue

• Theorem [Hartline & Roughgarden, 2009]: For independent regular distributions, VCG with bidder-specific reserve prices can guarantee 50% of the optimal revenue.
Approx. Optimal Auctions

• Still relies on knowing bidders’ distributions
  ➢ Can break down if the true distribution is different than the assumed distribution

• Theorem [Bulow and Klemperer, 1996]:
  For i.i.d. bidder valuations,
  \[ E[\text{Revenue of VCG with } n + 1 \text{ bidders}] \geq E[\text{Optimal revenue with } n \text{ bidders}] \]

• “Spend effort in getting one more bidder than in figuring out the optimal auction”
Simple Proof

• (n+1)-bidder VCG has the maximum expected revenue among all (n+1)-bidder DSIC auctions that always allocate the item
  ➢ Revenue Equivalence Theorem

• Consider the following (n+1)-bidder DSIC auction
  ➢ Run $n$-bidder Myerson on first $n$ bidders. If the item is unallocated, give it to agent $n + 1$ for free.
  ➢ As much expected revenue as $n$-bidder Myerson auction
  ➢ No more expected revenue than (n+1)-bidder VCG

• QED!
Optimizing Revenue is Hard

• Beyond single-parameter settings, the optimal auctions become even trickier

• **Example:** Two items, a single bidder with i.i.d. values for both items
  - **Q:** Shouldn’t the optimal auction just sell each item individually using Myerson’s auction?
  - **A:** No! Putting a take-it-or-leave-it offer on the two items bundled together can increase revenue!

• Slow progress on optimal auctions, but fast progress on simple and approximately optimal auctions
Mechanism Design
Without Money
Lack of Money

• Mechanism design with money:
  ➢ VCG can implement the welfare maximizing outcome because it can charge payments

• Mechanism design without money:
  ➢ Suppose you want to give away a single item, but cannot charge any payments
  ➢ Impossible to get meaningful information about valuations from strategic agents
  ➢ How would you maximize welfare as much as possible?
Lack of Money

• **One possibility:** Give the item to each of \( n \) bidders with probability \( 1/n \).

• Does not maximize welfare
  ➢ It’s impossible to maximize welfare without money

• Achieves an \( n \)-approximation of maximum welfare
  ➢ \( \max_v \frac{\max_i v_i}{(1/n) \sum_i v_i} \leq n \) (What is this?)

• Can’t do better than \( n \)-approximation
MD w/o Money Theme

1. Define the problem: agents, outcomes, valuations

2. Define the goal (e.g., maximizing social welfare)

3. Check if the goal can be achieved using a strategyproof mechanism
   ➢ “strategyproof” = DSIC

4. If not, find the strategyproof mechanism that provides the best approximation ratio
   ➢ Approximation ratio is similar to price of anarchy (PoA)
Facility Location

- Set of agents $N$
- Each agent $i$ has a true location $x_i \in \mathbb{R}$
- Mechanism $f$
  - Takes as input reports $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n)$
  - Returns a location $y \in \mathbb{R}$ for the new facility
- Cost to agent $i$: $c_i(y) = |y - x_i|$
- Social cost $C(y) = \sum_i c_i(y) = \sum_i |y - x_i|$
Facility Location

- Social cost $C(y) = \sum_i c_i(y) = \sum_i |y - x_i|$

- **Q:** Ignoring incentives, what choice of $y$ would minimize the social cost?

- **A:** The median location $\text{med}(x_1, ..., x_n)$
  - $n$ is odd $\rightarrow$ the unique $\text{"(n+1)/2"}^{\text{th}}$ smallest value
  - $n$ is even $\rightarrow$ “$n/2$”th or “$(n/2)+1$”st smallest value
  - **Why?**
Facility Location

• Social cost \( C(y) = \sum_i c_i(y) = \sum_i |y - x_i| \)

• Median is optimal (i.e., 1-approximation)

• What about incentives?

  ➢ Median is also strategyproof (SP)!

  ➢ Irrespective of the reports of other agents, agent \( i \) is best off reporting \( x_i \)
Median is SP

No manipulation can help
Max Cost

• A different objective function \( C(y) = \max_i |y - x_i| \)

• Q: Again ignoring incentives, what value of \( y \) minimizes the maximum cost?

• A: The midpoint of the leftmost \( \min_i x_i \) and the rightmost \( \max_i x_i \) locations (WHY?)

• Q: Is this optimal rule strategyproof?

• A: No! (WHY?)
Max Cost

• \( C(y) = \max_i |y - x_i| \)

• We want to use a strategyproof mechanism.

• **Question:** What is the approximation ratio of median for maximum cost?

  1. \( \in [1,2) \)
  2. \( \in [2,3) \)
  3. \( \in [3,4) \)
  4. \( \in [4,\infty) \)
Max Cost

• **Answer:** 2-approximation

• Other SP mechanisms that are 2-approximation
  - Leftmost: Choose the leftmost reported location
  - Rightmost: Choose the rightmost reported location
  - Dictatorship: Choose the location reported by agent 1
  - ...

Max Cost

• Theorem [Procaccia & Tennenholtz, ‘09]
  No deterministic SP mechanism has approximation ratio < 2 for maximum cost.

• Proof:
Max Cost + Randomized

• The Left-Right-Middle (LRM) Mechanism
  ➢ Choose $\min_i x_i$ with probability $\frac{1}{4}$
  ➢ Choose $\max_i x_i$ with probability $\frac{1}{4}$
  ➢ Choose $(\min_i x_i + \max_i x_i)/2$ with probability $\frac{1}{2}$

• Question: What is the approximation ratio of LRM for maximum cost?

• At most $\frac{(1/4) \cdot 2C + (1/4) \cdot 2C + (1/2) \cdot C}{C} = \frac{3}{2}$
Max Cost + Randomized

• Theorem [Procaccia & Tennenholtz, ‘09]: The LRM mechanism is strategyproof.

• Proof:

\[
\begin{align*}
\frac{1}{4} & \quad \frac{1}{2} & \quad \frac{1}{4} \\
2\delta & \quad \delta & \quad 2\delta
\end{align*}
\]
Max Cost + Randomized

• Exercise for you!
  Try showing that no randomized SP mechanism can achieve approximation ratio $< \frac{3}{2}$

• Suggested outline
  ➢ Consider two agents with $x_1 = 0$ and $x_2 = 1$
  ➢ Show that one of them has expected cost at least $\frac{1}{2}$
  ➢ What happens if that agent moves 1 unit farther from the other agent?