CSC304 Lecture 11

Mechanism Design w/ Money: Revenue maximization; Myerson’s auction
Announcements

• Returning graded midterm
  ➢ Was only able to keep my promise due to wonderful TAs

• Delighted by your performance!
  ➢ Given that the midterm was relatively hard

• Coming up: 4-5 questions of homework 2
Welfare vs Revenue

• In the auction setting...
  ➢ We choose an outcome $a$ based on agent valuations $\{v_i\}$
  ➢ And charge payments $p_i$ to each agent $i$

• In welfare maximization, we want to maximize $\sum_i v_i(a)$
  ➢ VCG = DSIC + maximizes welfare on every single instance
  ➢ Beautiful!

• In revenue maximization, we want to maximize $\sum_i p_i$
  ➢ We can still use DSIC mechanisms (revelation principle).
    BUT...
Welfare vs Revenue

- Different DSIC mechanisms are better for different instances.

- Example: 1 item, 1 bidder, unknown value $v$
  - DSIC = fix a price $r$, let the agent decide to “take it” ($v \geq r$) or “leave it” ($v < r$)
  - Maximize welfare $\rightarrow$ set $r = 0$
  - Maximize revenue $\rightarrow$ $r = ?$
    - Different $r$ are better for different $v$

- Must analyze in a Bayesian setting
Single-Item Auction Framework

• \( n \) bidders

• Value \( v_i \) of bidder \( i \) is drawn from distribution \( F_i \) with density \( f_i \) and support \([0, v_{\text{max}}]\)

• Principal knows \( \{F_i\} \), and wants to maximize \( E[\sum_i p_i] \)
  - Expectation over each \( v_i \) drawn i.i.d. from \( F_i \)
  - Principal wants to use a DSIC mechanism
    - IC part is without loss of generality (revelation principle)
    - Will see that can’t do better using BNIC mechanisms
Single Item, Single Bidder

• Revisiting 1 item, 1 bidder
• Value $v \sim F$
• Want to post a price $r$ on the item

• Revenue from price $r \rightarrow r \cdot (1 - F(r))$ (Why?)

• Awesome! Select $r^* = \arg\max_r r \cdot (1 - F(r))$
  ➢ “Monopoly price”
  ➢ Note: $r^*$ depends on $F$, but not on $v \Rightarrow$ DSIC
Single Item, Single Bidder

• Suppose the bidder’s value is drawn from the uniform distribution $U[0,1]$.
• Recall: $E[\text{Revenue}]$ from price $r$ is $r \cdot (1 - F(r))$

• $Q$: What is the optimal posted price?
• $Q$: What is the corresponding optimal revenue?

• Compare this to the revenue of VCG, which is 0
An Aside

• In welfare maximization, we are bound to always selling the item

• In revenue maximization, we are willing to risk leaving the item unsold
  ➢ If the item is not sold, you get 0 revenue
  ➢ But if sold, you can get more than 2\textsuperscript{nd} highest bid

• Subject to always selling the item, VCG actually has the highest revenue
  ➢ Revenue equivalence: “same allocation $\Rightarrow$ same payment”
Single Item, Two Bidders

- \( v_1, v_2 \sim U[0,1] \)

- VCG revenue = 2\(^{nd}\) highest bid = \( \min(v_1, v_2) \)
  - \( E[\min(v_1, v_2)] = 1/3 \)

- A possible improvement: “VCG with reserve price”
  - Reserve price \( r \).
  - Highest bidder gets the item if bid more than \( r \)
  - Pays \( \max(r, 2^{nd}\) highest bid)
Single Item, Two Bidders

• Reserve prices are ubiquitous
  ➢ E.g., opening bids in eBay auctions
  ➢ Guarantee a certain revenue to the auctioneer if item is sold

• $E[\text{revenue}] = E[\max(r, \min(v_1, v_2))]$
  ➢ Maximize over $r$?

• What about other DSIC mechanisms? What if there are more bidders? Other distributions?
Single-Parameter Environments

• Roger B. Myerson solved revenue optimal auctions in “single-parameter environments”

• Proposed a simple auction that maximizes expected revenue
Single-Parameter Environments

- Each agent $i$ has a private value $v_i \sim F_i$,
  - Value if the agent is “served”
  - **Example:** single-item auction $\rightarrow$ win the item
  - **Example:** combinatorial auction + single-minded bidder $\rightarrow$ get the desired set
  - Can potentially allow agents to be “fractionally” served

- Fixing bids of other agents...
  - Let $x_i(v_i) = \text{fraction served when reporting } v_i$
    - When fractional serving not allowed, this is in $\{0,1\}$
  - Let $p_i(v_i) = \text{payment charged when reporting } v_i$
Myerson’s Lemma

• Myerson’s Lemma:
  For a single-parameter environment, a strategy profile is in BNE under a mechanism if and only if
  1. $x_i(v_i)$ is monotone non-decreasing
  2. $p_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z) dz + p_i(0)$
    (typically, $p_i(0) = 0$)

• Intuition similar to 2nd price auction
  ➢ For every “δ” allocation, pay the lowest value that would have won it
Myerson’s Lemma

• Note: allocation determines unique payments

\[ p_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0) \]

• A corollary: revenue equivalence
  ➢ If two mechanisms use the same allocation \( x_i \), they “essentially” have the same expected revenue

• Another corollary: optimal revenue auctions
  ➢ Optimizing revenue = optimizing some function of allocation (easier to analyze)
Myerson’s Theorem

• “Expected Revenue = Expected Virtual Welfare”
  ➢ Recall: $p_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$
  ➢ Take expectation over draw of valuations + lots of calculus

$$E_{\{v_i \sim F_i\}}[\Sigma_i p_i(v_i)] = E_{\{v_i \sim F_i\}}[\Sigma_i \phi_i(v_i) \cdot x_i(v_i)]$$

• $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ is called the virtual value of bidder $i$
• Virtual welfare = sum of virtual values*allocations
Myerson’s Auction

• Need the allocation $x_i$ to be monotonic
• $E[\text{revenue}] = E[\text{virtual welfare}]$

• **Myerson’s auction**: “The auction that maximizes (expected) revenue is the one whose allocation maximizes the virtual welfare subject to monotonicity”

• Let’s apply this to some examples!
Example

• 2 bidders, 1 item, values drawn i.i.d. from $U[0,1]$

  ➢ $\varphi(v) = v - \frac{1-F(v)}{f(v)} = v - \frac{1-v}{1} = 2v - 1$

  ➢ Note: virtual value can be negative!!

• Given bids $(v_1, v_2), ...$

  ➢ Maximize $x_1 \cdot (2v_1 - 1) + x_2 \cdot (2v_2 - 1)$

  ➢ Subject to $x_1, x_2 \in \{0,1\}$ and $x_1 + x_2 \leq 1$
Optimal Auction Example

• Maximize $x_1 \cdot (2v_1 - 1) + x_2 \cdot (2v_2 - 1)$
  - $x_1, x_2 \in \{0,1\}$ and $x_1 + x_2 \leq 1$

• Prove on the board:
  - Allocation:
    - If $\exists$ bidder with value $\geq \frac{1}{2}$, give to the highest bidder.
    - If both have value $< \frac{1}{2}$, neither gets the item.
  - Payment if item sold = $\max(\frac{1}{2}, \text{lesser bid})$

• Precisely VCG with reserve price $\frac{1}{2}$
Optimal Auctions

• **Theorem:** For a single item and $n$ bidders whose valuations are drawn i.i.d., the optimal auction is VCG with reserve price $\varphi^{-1}(0)$.
  - Note: Reserve price is independent of #bidders!

• *Wait!* We didn’t check for monotonicity of allocation!

• It turns out that for “nice” distributions, maximizing virtual welfare already gives a monotonic allocation rule!
Special Distributions

• **Regular Distributions:**
  A distribution $F$ is regular if its virtual value function $\nu - (1 - F(\nu))/f(\nu)$ is non-decreasing.

• **Lemma:** If all $F_i$’s are regular, the virtual welfare maximizing rule is monotone.

• **Monotone Hazard Rate (MHR):**
  A distribution $F$ has monotone hazard rate if $(1 - F(\nu))/f(\nu)$ is non-increasing.
  ➢ Important special case (MHR $\Rightarrow$ Regular)
Special Distributions

• Not crazy assumptions

➢ Many practical distributions are MHR: e.g., uniform, exponential, Gaussian.

➢ Some important distributions are not MHR, but still regular: e.g., power-law distributions.
Optimal Single-Item Auction

• **Allocation**: Give the item to agent $i$ with highest $\phi_i(v_i)$ if that is non-negative

• **Payment**: “lowest bid that still would have won”
  - Follows from $p_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$

• **All $F_i$’s are equal to $F$ and regular**:
  - $r^*$ = monopoly price of $F$
  - Item goes to the highest bidder if bid more than $r^*$
  - Payment charged is $\max(r^*, 2\text{nd highest bid})$,
  - VCG with reserve price $r^*$!
Extensions

• Irregular distributions:
  - E.g., multi-modal or extremely heavy tail distributions
  - Need to add the monotonicity constraint
  - Turns out, we can “iron” irregular distributions to make them regular, and use standard Myerson’s framework

• Relaxing DSIC to BNIC
  - Myerson’s mechanism has optimal revenue among all DSIC mechanisms
  - Turns out, it also has optimal revenue among the much larger class of BNIC mechanisms!
Approx. Optimal Auctions

• For i.i.d. regular distributions, the optimal auction is simple (VCG with reserve price)

• For unequal distributions, it can be very complex
  ➢ In practice, we prefer simple auctions that bidders can understand, but still want approximately optimal revenue

• Theorem [Hartline & Roughgarden, 2009]: For independent regular distributions, VCG with bidder-specific reserve prices is a 2-approximation of the optimal revenue.
Approximately Optimal

• Still relies on knowing bidders’ distributions
  ➢ Dangerous! Guarantees can break down if the true distribution is different from the assumed distribution

• Theorem [Bulow and Klemperer, 1996]: For i.i.d. bidder valuations,
  \[ E[\text{Revenue of VCG with } n + 1 \text{ bidders}] \geq E[\text{Optimal revenue with } n \text{ bidders}] \]

• “Spend effort in getting one more bidder than in figuring out the optimal auction”
Simple proof

• VCG with $n + 1$ bidders has the maximum revenue among all $n + 1$ bidder DSIC auctions that always allocate the item [via revenue equivalence]

• Consider the auction: “Run $n$-bidder Myerson on first $n$ bidders. If the item is unallocated, give it to agent $n + 1$ for free.”
  - $n + 1$ bidder DSIC auction
  - As much revenue as $n$-bidder Myerson auction
Optimizing Revenue is Hard

• Slow progress beyond single-parameter setting
  ➢ Even with just two items and one bidder with i.i.d. values for both items, the optimal auction **DOES NOT** run Myerson’s auction on individual items!
  ➢ “Take-it-or-leave-it” offers for the two items bundled might increase revenue

• But nowadays, the focus is on simple, approximately optimal auctions instead of complicated, optimal auctions.