CSC373

Review

Topics

- Divide and conquer
- Greedy algorithms
- Dynamic programming
- Network flow
- Linear programming
- Complexity
- Approximation algorithms

Greedy Algorithms

- Greedy algorithm outline
  - We want to find the optimal solution maximizing some objective f over a large space of feasible solutions
  - Solution x is composed of several parts (e.g. a set)
  - Instead of directly computing x…
    - Consider one element at a time in some greedy ordering
    - Make a decision about that element before moving on to future elements (and without knowing what will happen for the future elements)

Greedy Algorithms

- Proof of optimality outline
  - Strategy 1:
    - G_i = greedy solution after i steps
    - Show that $\forall i$, there is some optimal solution OPT_i s.t. $G_i \subseteq OPT_i$
      - "Greedy solution is promising"
    - By induction
    - Then the final solution returned by greedy must be optimal
  - Strategy 2:
    - Same as strategy 1, but more direct
    - Consider OPT that matches greedy solution for as many steps as possible
    - If it doesn't match in all steps, find another OPT which matches for one more step (contradiction)

Dynamic Programming

- Key steps in designing a DP algorithm
  - "Generalize" the problem first
    - E.g. instead of computing edit distance between strings $X = x_1, \ldots, x_m$ and $Y = y_1, \ldots, y_n$, we compute $E[i, j]$ = edit distance between i-prefix of X and j-prefix of Y for all (i, j)
    - The right generalization is often obtained by looking at the structure of the "subproblem" which must be solved optimally to get an optimal solution to the overall problem
  - Remember the difference between DP and divide-and-conquer
  - Sometimes you can save quite a bit of space by only storing solutions to those subproblems that you need in the future

Dynamic Programming

- Dynamic programming applies well to problems that have optimal substructure property
  - Optimal solution to a problem contains (or can be computed easily given) optimal solution to subproblems.

- Recall: divide-and-conquer also uses this property
  - You can think of divide-and-conquer as a special case of dynamic programming, where the two (or more) subproblems you need to solve don't "overlap"
  - So there's no need for memoization
  - In dynamic programming, one of the subproblems may in turn require solution to the other subproblem…

Dynamic Programming

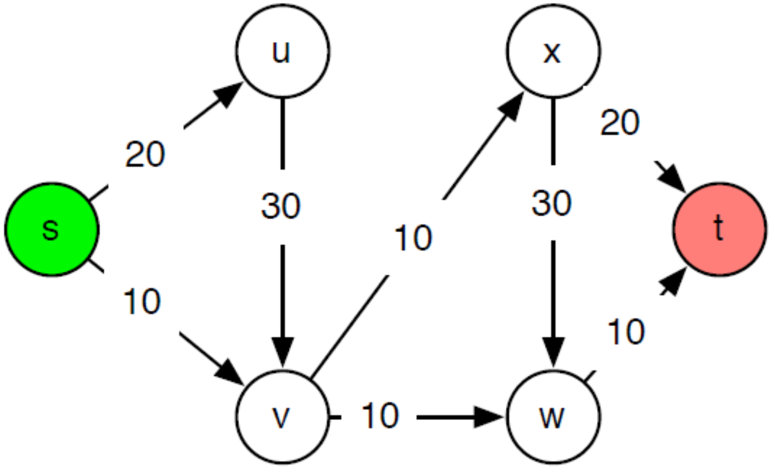
- Top-Down may be preferred…
  - …when not all sub-solutions need to be computed on some inputs
  - …because one does not need to think of the "right order" in which to compute sub-solutions

- Bottom-Up may be preferred…
  - …when all sub-solutions will anyway need to be computed
  - …because it is sometimes faster as it prevents recursive call overheads and unnecessary random memory accesses

Network Flow

- Input
  - A directed graph $G = (V, E)$
  - Edge capacities $c : E \to \mathbb{R}_{\geq 0}$
  - Source node s, target node t

- Output
  - Maximum "flow" from s to t



Ford-Fulkerson Algorithm

```
MaxFlow(G):
  // initialize:
  Set f(e) = 0 for all e in G

  // while there is an s-t path in G_f:
  While P = FindPath(s,t,Residual(G,f))!=None:
    f = Augment(f,P)
    UpdateResidual(G,f)
  EndWhile
  Return f
```

Max Flow - Min Cut

- Theorem: In any graph, the value of the maximum flow is equal to the capacity of the minimum cut.

- Ford-Fulkerson can be used to find the min cut
  - Find the max flow f^*
  - Let $A^* =$ set of all nodes reachable from s in residual graph G_{f^*}
    - Easy to compute using BFS
  - Then $(A^*, V \setminus A^*)$ is min cut

LP, Standard Formulation

- Input: $c, a_1, a_2, \dots, a_m \in \mathbb{R}^n, b \in \mathbb{R}^m$
  - There are n variables and m constraints
- Goal:

$$\text{Maximize } c^T x$$

$$\text{Subject to } a_1^T x \leq b_1$$

$$a_2^T x \leq b_2$$

$$\vdots$$

$$a_m^T x \leq b_m$$

$$x \geq 0$$

n variables

m constraints

n more constraints

LP, Standard Matrix Form

- Input: $c, a_1, a_2, \dots, a_m \in \mathbb{R}^n, b \in \mathbb{R}^m$
  - There are n variables and m constraints
- Goal:

$$\text{Maximize } c^T x$$

$$\text{Subject to } Ax \leq b$$

$$x \geq 0$$

n variables

m constraints

n more constraints

Convert to Standard Form

- What if the LP is not in standard form?
  - Constraints that use $\geq$
    - $a^T x \geq b \;\; \Leftrightarrow \;\; -a^T x \leq -b$
  - Constraints that use equality
    - $a^T x = b \;\; \Leftrightarrow \;\; a^T x \leq b, \;\; a^T x \geq b$
  - Objective function is a minimization
    - Minimize $c^T x$ $\Leftrightarrow$ Maximize $-c^T x$
  - Variable is unconstrained
    - x with no constraint $\Leftrightarrow$ Replace x by two variables $x'$and x'', replace every occurrence of x with $x' - x''$, and add constraints $x' \geq 0, \;\; x'' \geq 0$

Duality

Primal LP	**Dual LP**
$\max\ \mathbf{c}^T\mathbf{x}$	$\min\ \mathbf{y}^T\mathbf{b}$
$\mathbf{A}\mathbf{x} \leq \mathbf{b}$	$\mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

- Weak duality theorem:
  - For any primal feasible x and dual feasible y, $c^T x \leq y^T b$
- Strong duality theorem:
  - For any primal optimal x^* and dual optimal y^*, $c^T x^* = (y^*)^T b$

P

- P (polynomial time)
  - The class of all decision problems computable by a TM in polynomial time

NP

- NP (nondeterministic polynomial time)
  - The class of all decision problems for which a YES answer can be verified by a TM in polynomial time given polynomial length "advice" or "witness".
  - There is a polynomial-time verifier TM V and another polynomial p such that
    - For all YES inputs x, there exists y with $|y| = p(|x|)$ on which $V(x, y)$ returns YES
    - For all NO inputs x, $V(x, y)$ returns NO for every y
  - Informally: "Whenever the answer is YES, there's a short proof of it."

co-NP

- co-NP
  - Same as NP, except whenever the answer is NO, we want there to be a short proof of it

Reductions

- Problem A is p-reducible to problem B if an "oracle" (subrouting) for B can be used to efficiently solve A
  - You can solve A by making polynomially many calls to the oracle and doing additional polynomial computation

NP-completeness

- NP-completeness
  - A problem B is NP-complete if it is in NP and every problem A in NP is p-reducible to B
  - Hardest problems in NP
  - If one of them can be solved efficiently, every problem in NP can be solved efficiently, implying P=NP

- Observation:
  - If A is in NP, and some NP-complete problem B is p-reducible to A, then A is NP-complete too
    - "If I could solve A, then I could solve B, then I could solve any problem in NP"

Review of Reductions

- If you want to show that problem B is NP-complete
- Step 1: Show that B is in NP
  - Some polynomial-size advice should be sufficient to verify a YES instance in polynomial time
  - No advice should work for a NO instance

  - Usually, the solution of the “search version” of the problem works
    - But sometimes, the advice can be non-trivial
    - For example, to check LP optimality, one possible advice is the values of both primal and dual variables, as we saw in the last lecture

Review of Reductions

- If you want to show that problem B is NP-complete
- Step 2: Find a known NP-complete problem A and reduce it to B (i.e. show A $\leq_p$ B)
  - This means taking an arbitrary instance of A, and solving it in polynomial time using an oracle for B
    - Caution 1: Remember the direction. You are “reducing known NP-complete problem to your current problem”.
    - Caution 2: The size of the B-instance you construct should be polynomial in the size of the original A-instance
  - This would show that if B can be solved in polynomial time, then A can be as well
  - Some reductions are trivial, some are notoriously tricky…

Approximation Algorithms

- We focus on optimization problems
  - Decision problem: "Is there…where…≥ k?"
    - E.g. "Is there an assignment which satisfies at least k clauses of a given formula φ?"
  - Optimization problem: "Find…which maximizes…"
    - E.g. "Find an assignment which satisfies the maximum possible number of clauses from a given formula φ."
  - Recall that if the decision problem is hard, then the optimization problem is hard too

Approximation Algorithms

- There is a function $Profit$ we want to maximize or a function $Cost$ we want to minimize

- Given input instance I…
  - Our algorithm returns a solution $ALG(I)$
  - An optimal solution maximizing $Profit$ or minimizing $Cost$ is $OPT(I)$
  - Then, the approximation ratio of ALG on instance I is

$$\frac{Profit\left(OPT(I)\right)}{Profit\left(ALG(I)\right)} \quad \text{or} \quad \frac{Cost\left(ALG(I)\right)}{Cost\left(OPT(I)\right)}$$

Approximation Algorithms

- Approximation ratio of ALG on instance I is

$$\frac{Profit(OPT(I))}{Profit(ALG(I))} \text{ or } \frac{Cost(ALG(I))}{Cost(OPT(I))}$$

  - Note: These are defined to be ≥ 1 in each case.
    - 2-approximation = half the optimal profit / twice the optimal cost

- ALG has worst-case c-approximation if for each instance I…

$$Profit(ALG(I)) \geq \frac{1}{c} \cdot Profit(OPT(I)) \; or$$

$$Cost(ALG(I)) \leq c \cdot Cost(OPT(I))$$

Techniques

- Greedy algorithms
- LP relaxation => rounding
- Local search