

CSC373

Complexity

Introduction to Complexity

- You have a problem at hand
- You try every technique known to humankind for finding a polynomial time algorithm but fail.
- You try every technique known to humankind for proving that there cannot exist a polynomial time algorithm for your problem but fail.
- **What do you do?**
 - Prove that it is NP-complete, of course!

Turing Machines

- “Which problems can a computer (not) solve in a certain amount of time?”
 - How do we mathematically define what a computer is?
- Alan Turing (“Father of Computer Science”), 1936
 - Introduced a mathematical model
 - “Turing machine”
 - All present-day computers can be simulated by a Turing machine
 - **Fun fact:** TMs can simulate quantum computers too, just inefficiently

Turing Machines

- We won't formally introduce...but at a high level...
- **Turing machine**
 - **Tape**
 - Input is given on tape
 - Intermediate computations can be written there
 - Output must be written there
 - **Head pointer**
 - Initially pointing at start of input on tape
 - Maintains an internal **“state”**
 - A **transition function** describes how to change state, move head pointer, and read/write symbols on tape

Computability

- Church-Turing Thesis
 - “Everything that is computable can be computed by a Turing machine”
 - Widely accepted, not “proven”
 - There are problems which a Turing machine cannot solve, regardless of the amount of time available
 - E.g., the halting problem
- What about the problems we *can* solve? How do we define the time required?
 - Need to define an encoding of the input and output

Encoding

- **What can we write on the tape?**
 - S = a set of finite symbols
 - $S^* = \bigcup_{n \geq 0} S^n$ = set of all finite strings using symbols from S
- **Input:** $w \in S^*$
 - Length of input = $|w|$ = length of w on tape
- **Output:** $f(w) \in S^*$
 - Length of output = $|f(w)|$
 - **Decision problems:** output = “YES” or “NO”
 - E.g., “does there exist a flow of value at least 7 in this network?”

Encoding

- **Example:**

- “Given a_1, a_2, \dots, a_n , compute $\sum_{i=1}^n a_i$ ”

- Suppose we are told that $a_i \leq C$ for all i

- **What $|S|$ should we use?**

- $S = \{0,1\}$ ($|S| = 2$, binary representation)

- Length of input = $O(\log_2 a_1 + \dots + \log_2 a_n) = O(n \log_2 C)$

- What about 3-ary ($|S| = 3$) or 18-ary ($|S| = 18$)?

- Only changes the length by a constant factor, still $O(n \log C)$

- What about unary (conceptually, $|S| = 1$)?

- Length blows up exponentially to $O(nC)$

- Binary is already good enough, but unary isn't

Efficient Computability

- **Polynomial-time computability**

- A TM solves a problem in polynomial time if there is a polynomial p such that on every instance of n -bit input and m -bit output, the TM halts in at most $p(n, m)$ steps
- Polynomial: $n, n^2, 5n^{100} + 1000n^3, n \log^{100} n = o(n^{1.001})$
- Non-polynomial: $2^n, 2^{\sqrt{n}}, 2^{\log^2 n}$

- **Extended Church-Turing Hypothesis**

- “Everything that is efficiently computable is computable by a TM in polynomial time”
- Much less widely accepted, especially today
- But in this course, **efficient = polynomial-time**

P

- P (polynomial time)

- The class of all decision problems computable by a TM in polynomial time

- Examples

- Addition, multiplication, square root
- Shortest paths
- Network flow
- Fast Fourier transform
- Checking if a given number is a prime [Agrawal-Kayal-Saxena 2002]
- ...

NP

- NP (nondeterministic polynomial time) intuition
 - Subset sum problem:
Given an array $\{-7, -3, -2, 5, 8\}$, is there a zero-sum subset?
 - Enumerating all subsets is exponential
 - But...given $\{-3, -2, 5\}$, we can verify in polynomial time that it is indeed a valid subset and has zero sum
 - A nondeterministic Turing machine could “guess” the solution and then test if it has zero sum in polynomial time

NP

- **NP (nondeterministic polynomial time)**
 - The class of all decision problems for which a YES answer can be verified by a TM in polynomial time given polynomial length “advice” or “witness”.
 - There is a polynomial-time verifier TM V and another polynomial p such that
 - For all YES inputs x , there exists advice y with $|y| = p(|x|)$ on which $V(x, y)$ returns YES
 - For all NO inputs x , $V(x, y)$ returns NO for every possible y
 - **Informally:**
 - When the answer is YES, there is a short proof of it.
 - When the answer is NO, there is no short proof of it.

co-NP

- **co-NP**
 - Same as NP, except whenever the answer is NO, there is a short proof of it
- **Open questions**
 - NP = co-NP?
 - P = NP \cap co-NP?
 - And...drum roll please...

$$P = NP?$$

P versus NP

- Lance Fortnow in his article on P and NP in Communications of the ACM, Sept 2009

“The P versus NP problem has gone from an interesting problem related to logic to perhaps the most fundamental and important mathematical question of our time, whose importance only grows as computers become more powerful and widespread.”

Millenium Problems

- Award of \$1M for each problem by the Clay Math institute

1. Birch and Swinnerton-Dyer Conjecture

2. Hodge Conjecture

3. Navier-Stokes Equations

4. $P = NP?$

Claim: Worth \gg \$1M

5. Poincare Conjecture (Solved)¹

6. Riemann Hypothesis

7. Yang-Mills Theory

¹Solved by Grigori Perelman (2003): Prize unclaimed

Cook's Conjecture

- Cook's conjecture
 - (And every sane person's belief...)
 - P is likely not equal to NP
- Why do we believe this?
 - There is a large class of problems (NP-complete)
 - By now, contains thousands and thousands of problems
 - Each problem is the "hardest problem in NP"
 - If you can efficiently solve *any one of them*, you can efficiently solve *every problem in NP*
 - Despite decades of effort, no polynomial time solution has been found for *any of them*

Reductions

- Decision problem A is **p-reducible** to decision problem B (denoted $A \leq_p B$) if there exists a polynomial-time TM that can convert any instance of A into an instance of B with the same answer
 - An oracle for solving B efficiently can be used to solve A efficiently!
- **Question:** If A is p-reducible to B , then which of the following is true?
 - a) If A cannot be solved efficiently, then neither can B .
 - b) If B cannot be solved efficiently, then neither can A .
 - c) Both.
 - d) None.

Reductions

- Decision problem A is **p-reducible** to decision problem B (denoted $A \leq_p B$) if there exists a polynomial-time TM that can convert any instance of A into an instance of B with the same answer
 - An oracle for solving B efficiently can be used to solve A efficiently!
- **Question:** If I want to prove that my problem X is “hard”, I should:
 - a) Reduce my problem X to a known hard problem.
 - b) Reduce a known hard problem to my problem X .
 - c) Both.
 - d) None.

NP-completeness

- NP-completeness

- A problem B is NP-complete if it is in NP and **every** problem A in NP is p-reducible to B
- Hardest problems in NP
- If one of them can be solved efficiently, every problem in NP can be solved efficiently, implying $P=NP$

- Observation:

- If A is in NP, and some NP-complete problem B is p-reducible to A , then A is NP-complete too
 - Every problem in NP $\leq_p B \leq_p A$

NP-completeness

- But this uses an already known NP-complete problem to prove another problem is NP-complete
- **How do we find the *first* NP-complete problem?**
 - How do we know there are *any* NP-complete problems at all?
 - Key result by Cook
 - First NP-complete problem: SAT
 - By a direct reduction from every problem in NP to SAT
 - “From first principles”

CNF Formulas

- **Conjunctive normal form (CNF)**

- Boolean **variables** x_1, x_2, \dots, x_n
- Their **negations** $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$
- **Literal** ℓ : a variable or its negation
- **Clause** $C = \ell_1 \vee \ell_2 \vee \dots \vee \ell_r$ is a disjunction of literals
- **CNF formula** $\varphi = C_1 \wedge C_2 \wedge \dots \wedge C_m$ is a conjunction of clauses
 - **k CNF**: Each clause has at most k literals
 - **Exact k CNF**: Each clause has exactly k literals
- Example of (Exact) 3CNF

$$\varphi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4) \wedge (\bar{x}_3 \vee \bar{x}_4 \vee x_1)$$

SAT and Exact 3SAT

- Example of (Exact) 3CNF

$$\varphi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4) \wedge (\bar{x}_3 \vee \bar{x}_4 \vee x_1)$$

- “SAT” (Satisfiability) Problem:

- A CNF formula φ is **satisfiable** if there is an assignment of truth values (TRUE/FALSE) to variables under which the formula evaluates to TRUE
 - That means, in each clause, at least one literal is TRUE
- SAT: “Given a CNF formula φ , is it satisfiable?”
- Exact 3SAT: “Given an exact 3CNF formula φ , is it satisfiable?”

SAT and Exact 3SAT

- **Cook-Levin Theorem**
 - SAT (and even Exact 3SAT) is NP-complete
- Doesn't use any known NP-complete problem
 - Directly reduces any given NP problem to SAT
 - Reduction is a bit complex, so we'll defer it until later
 - **But for now, let's assume SAT and Exact 3SAT are NP-complete and reduce them to other problems (and then those problems to other problems...)**

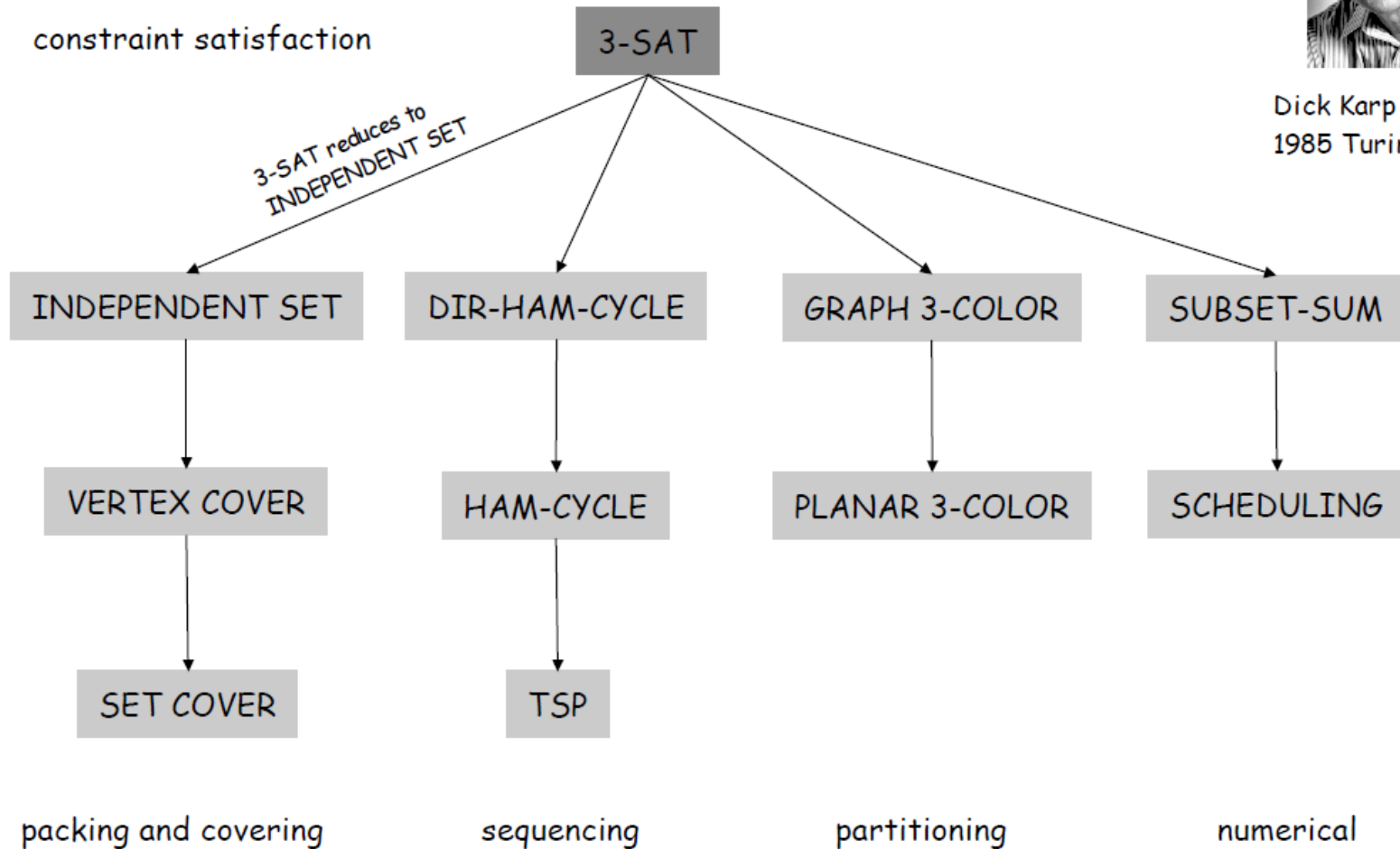
Just A Tad Bit of History

- [Cook 1971]
 - Proved Exact 3SAT is NP-complete in seminal paper
- [Karp 1972]
 - Showed that 20 other problems are also NP-complete
 - “Karp's 21 NP-complete problems”
 - Renewed interest in this idea
- 1982: Cook wins the Turing award
- 1985: Karp wins the Turing award

Polynomial-Time Reductions

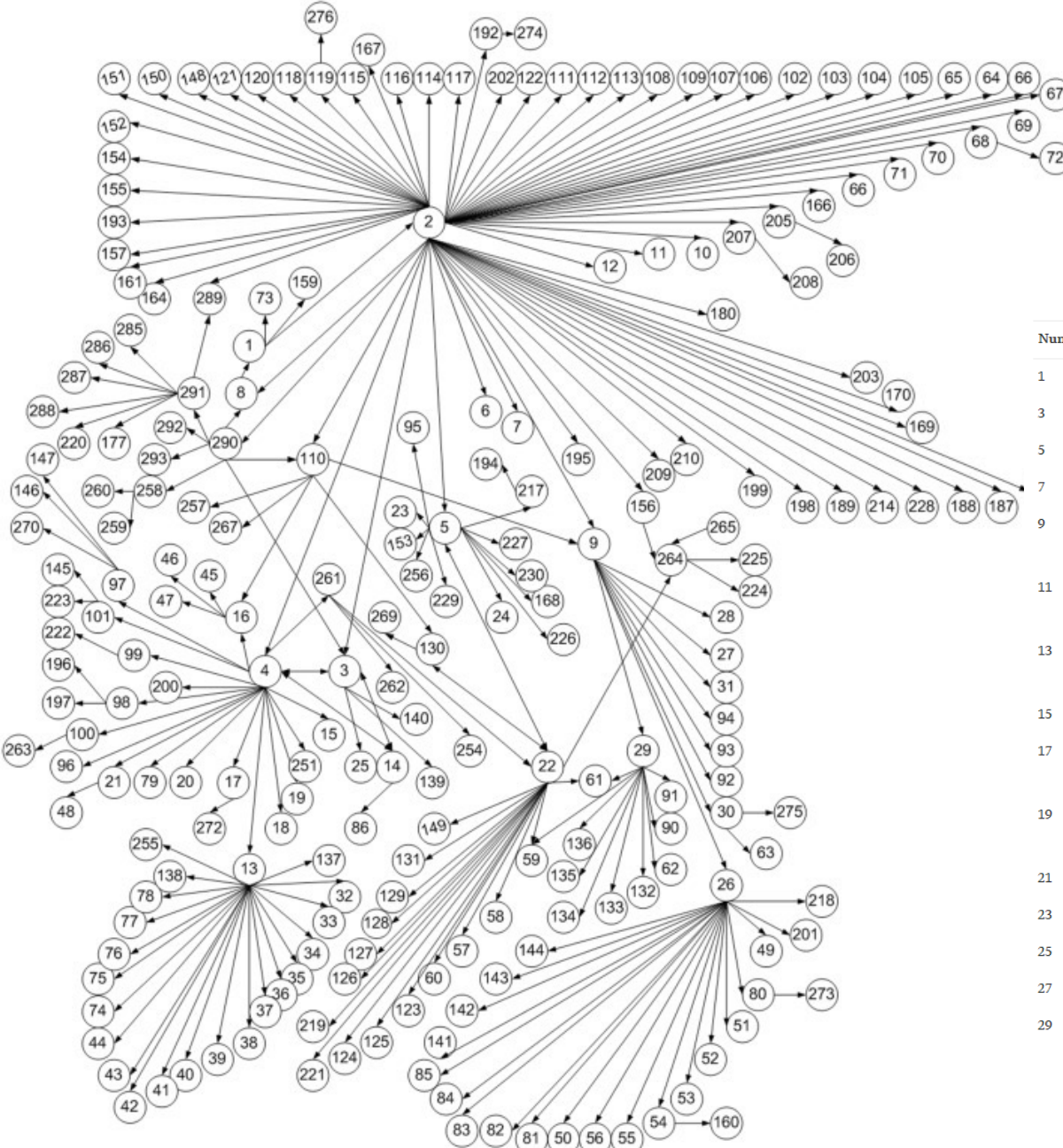


Dick Karp (1972)
1985 Turing Award



Survey of polynomial transformations between NP-complete problems

Jorge A. Ruiz-Vanoye ^a, Joaquín Pérez-Ortega ^b, Rodolfo A. Pazos R. ^c, Ocotlán Díaz-Parra ^d, Juan Frausto-Solís ^e, Hector J. Fraire Huacuja ^c, Laura Cruz-Reyes ^c, José A. Martínez F. ^c

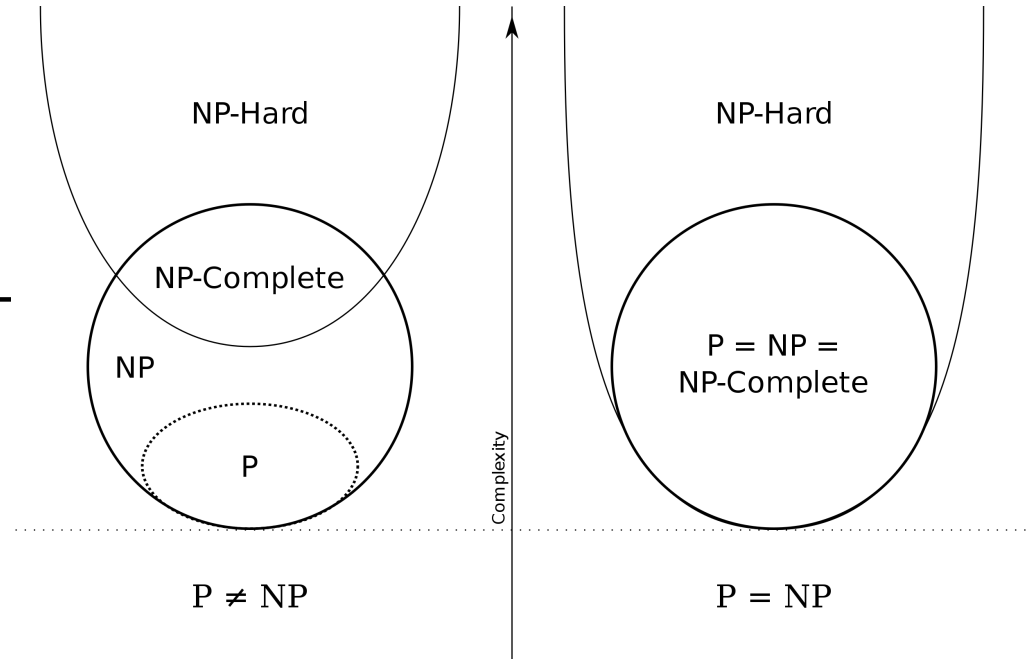
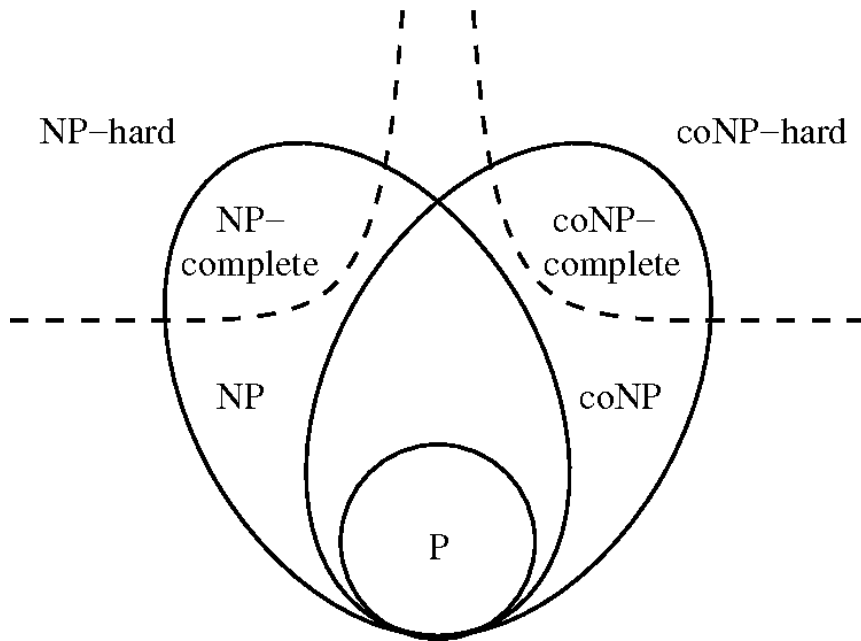


Number	Name of problem	Number	Name of problem
1	Satisfiability (SAT)	2	3-Satisfiability (3SAT)
3	Clique (clique cover)	4	Vertex cover
5	Subset sum	6	Hitting string
7	Chinese postman for mixed graphs	8	Graph colorability
9	Three-Dimensional matching (3DM)	10	Rectilinear picture compression
11	Tableau equivalence	12	Consistency of databases frequency tables
13	Hamiltonian Circuit (Directed Hamiltonian path, Undirected Hamiltonian path)	14	Independent set
15	Setbasis	16	Hitting set
17	Comparative containment	18	Multiple copy file allocation
19	Shortest common supersequence	20	Longest common subsequence
21	Minimum cardinality key	22	Partition
23	K'th largest subset	24	Capacity assignment
25	Conjunctive Boolean query	26	Exact cover by 3-sets (X3C)
27	Minimum test set	28	3-Matroid intersection
29	3-Partition	30	Numerical three-dimensional matching

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Complexity



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Reductions

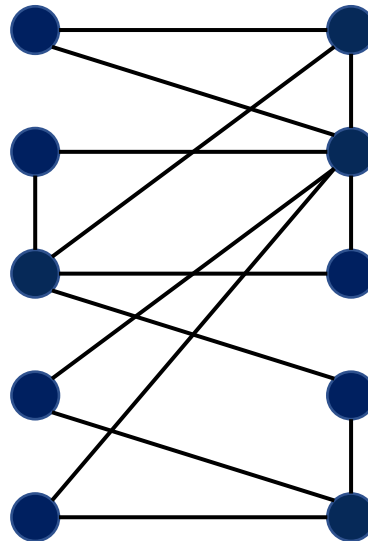
Independent Set

- **Problem**

- **Input:** Undirected graph $G = (V, E)$, integer k
- **Question:** Does there exist a subset of vertices $S \subseteq V$ with $|S| = k$ such that for each edge, **at most one** of its endpoints is in S ?

Example:

- Does this graph have an independent set of size 6?
 - Yes!
- Does this graph have an independent set of size 7?
 - No!



● = independent set

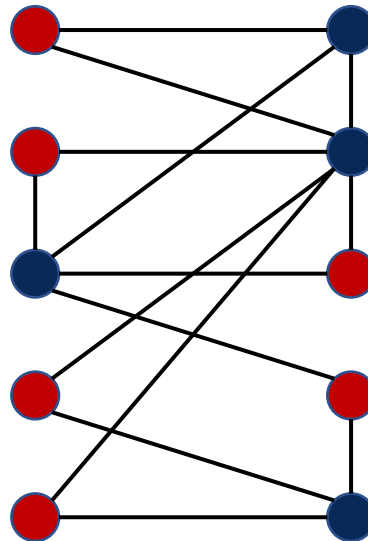
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Independent Set

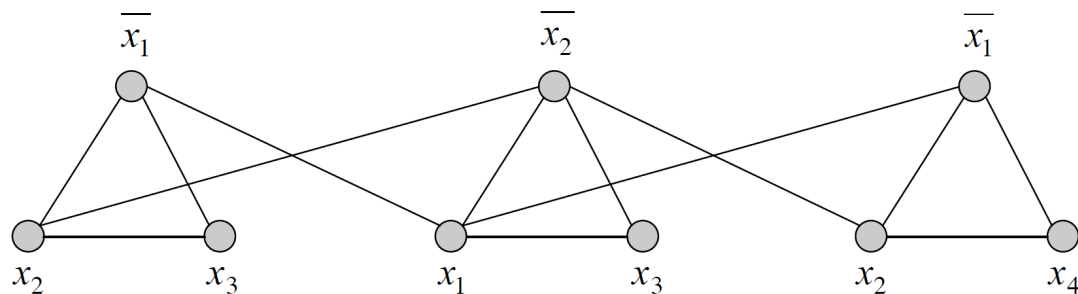
- Claim: Independent Set is in NP

- Recall: We need to show that there is a polynomial-time algorithm which
 - Can accept every YES instance with the right polynomial-size advice
 - Will not accept a NO instance with any advice
- **Advice:** the actual independent set S
- **Algorithm:** check if S is an independent set and if $|S| = k$
- Simple!

Independent Set

• Claim: Exact 3SAT \leq_p Independent Set

- Given a formula φ of Exact 3SAT with k clauses, construct an instance (G, k) of Independent Set as follows
 - Create 3 vertices for each clause (one for each literal)
 - Connect them in a triangle
 - Connect the vertex of each literal to each of its negations

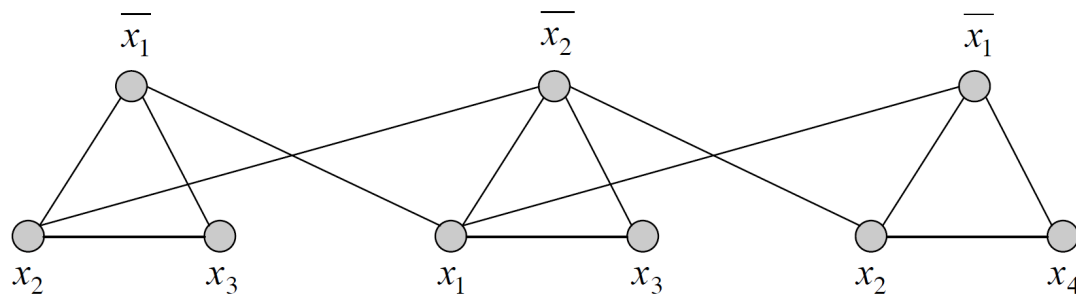


$$\Phi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4)$$

Independent Set

➤ Why does this work?

- Exact 3SAT = YES \Rightarrow Independent Set = YES
 - From each clause, take any literal that is TRUE in the assignment
- Independent Set = YES \Rightarrow Exact 3SAT = YES
 - Independent set S must contain one vertex from each triangle
 - No literal and its negation are both in S
 - Set literals in S to TRUE, their negations to FALSE, and the rest to arbitrary values



$$\Phi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4)$$

Subset Sum

- **Problem**

- **Input:** Set of integers $S = \{w_1, \dots, w_n\}$, integer W
- **Question:** Is there $S' \subseteq S$ that adds up to exactly W ?

- **Example**

- $S = \{1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344\}$, $W = 3754$?
- Yes!
 - $1 + 16 + 64 + 256 + 1040 + 1093 + 1284 = 3754$

Subset Sum

- **Claim: Subset Sum is in NP**

- Recall: We need to show that there is a polynomial-time algorithm which
 - Can accept every YES instance with the right polynomial-size advice
 - Will not accept a NO instance with any advice
- **Advice:** the actual subset S'
- **Algorithm:** check that S' is indeed a subset of S and sums to W
- Simple!

Subset Sum

- Claim: $\text{Exact 3SAT} \leq_p \text{Subset Sum}$

- Given a formula φ of Exact 3SAT, we want to construct (S, W) of Subset Sum with the same answer
- In the table in the following slide:
 - Columns are for variables and clauses
 - Each row is a number in S , represented in decimal
 - Number for literal ℓ : has 1 in its variable column and in the column of every clause where that literal appears
 - Number selected = literal set to TRUE
 - “Dummy” rows: can help make the sum in a clause column 4 if and only if at least one literal is set to TRUE

Subset Sum

Decimal representation

- Claim: $\text{Exact 3SAT} \leq_p \text{Subset Sum}$

$$C_1 = \bar{x} \vee y \vee z$$

$$C_2 = x \vee \bar{y} \vee z$$

$$C_3 = \bar{x} \vee \bar{y} \vee \bar{z}$$

dummies to get clause columns to sum to 4

	x	y	z	C ₁	C ₂	C ₃
x	1	0	0	0	1	0
¬ x	1	0	0	1	0	1
y	0	1	0	1	0	0
¬ y	0	1	0	0	1	1
z	0	0	1	1	1	0
¬ z	0	0	1	0	0	1
	0	0	0	1	0	0
	0	0	0	2	0	0
	0	0	0	0	1	0
	0	0	0	0	2	0
	0	0	0	0	0	1
	0	0	0	0	0	2
W	1	1	1	4	4	4

Subset Sum

- Note

- The Subset Sum instance we constructed has “large” numbers
 - Their values are exponentially large ($\sim 10^{\#variables + \#clauses}$)
 - But the number of bits required to write them is polynomial
- Can we hope to construct Subset Sum instance with numbers whose values are only $poly(\#variables, \#clauses)$ large?
 - Unlikely, as that would prove $P = NP!$
 - Like Knapsack, Subset Sum can be solved in *pseudo-polynomial time* (i.e., in polynomial time if the numbers are only polynomially large in value)

Review of Reductions

- If you want to show that problem B is NP-complete
- **Step 1: Show that B is in NP**
 - Some polynomial-size advice should be sufficient to verify a YES instance in polynomial time
 - No advice should work for a NO instance
 - Usually, the solution of the “search version” of the problem works
 - But sometimes, the advice can be non-trivial
 - For example, to **check LP optimality**, one possible advice is the **values of both primal and dual variables**, as we saw in the last lecture

Review of Reductions

- If you want to show that problem B is NP-complete
- **Step 2: Find a known NP-complete problem A and reduce it to B (i.e., show $A \leq_p B$)**
 - This means taking an arbitrary instance of A, and solving it in polynomial time using an oracle for B
 - Caution 1: Remember the direction. You are “reducing known NP-complete problem to your current problem”.
 - Caution 2: The size of the B-instances you construct should be polynomial in the size of the original A-instance
 - This would show that if B can be solved in polynomial time, then A can be as well
 - Some reductions are trivial, some are notoriously tricky...

Binary Integer Linear Programming (BILP)

- **Problem**

- **Input:** $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}, k \in \mathbb{R}$

- **Question:** Does there exist $x \in \{0,1\}^n$ such that $c^T x \geq k$ and $Ax \leq b$?

- Decision variant of “maximize $c^T x$ subject to $Ax \leq b$ ” but instead of any $x \in \mathbb{R}^n$ with $x \geq 0$, we are restricting x to binary.

- Does restricting search space make the problem easier or harder?
 - This is actually NP-complete!

BILP Feasibility

- An even simpler problem
 - Special case where $c = k = 0$, so $c^T x \geq k$ is always true

- **Problem**

- **Input:** $b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$
 - **Question:** Does there exist $x \in \{0,1\}^n$ such that $Ax \leq b$?
-
- Just need to find a feasible solution
 - This is still NP-complete!

BILP Feasibility

- Claim: BILP Feasibility is in NP

- Recall: We need to show that there is a polynomial-time algorithm which
 - Can accept every YES instance with the right polynomial-size advice
 - Will not accept a NO instance with any advice
- **Advice:** simply a vector x satisfying $Ax \leq b$
- **Algorithm:** Check if $Ax \leq b$
- Simple!

BILP Feasibility

- Claim: Exact 3SAT \leq_p BILP Feasibility

- Take any formula φ of Exact 3SAT
- Create a binary variable x_i for each variable x_i in φ
 - We'll represent its negation \bar{x}_i with $1 - x_i$
- For each clause C , we want at least one of its three literals to be TRUE
 - Just make sure their sum is at least 1
 - E.g., $C = x_1 \vee \bar{x}_2 \vee \bar{x}_3 \Rightarrow x_1 + (1 - x_2) + (1 - x_3) \geq 1$
- Easy to check that
 - this is a polynomial reduction
 - Resulting system has a feasible solution if and only if φ is satisfiable

ILP Feasibility

- **Problem**

- **Input:** $b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$

- **Question:** Does there exist $x \in \mathbb{Z}^n$ such that $Ax \leq b$?

- To prove that this is NP-hard, there is an obvious reduction from BILP feasibility to ILP feasibility

- What about membership in NP?

- **Advice:** simply a vector x satisfying $Ax \leq b$

- **Algorithm:** Check if $Ax \leq b$

- Simple?

- No, not clear if, in every YES instance, there's a polynomial-length "advice" vector x satisfying $Ax \leq b$

On the Complexity of Integer Programming

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ABSTRACT. A simple proof that integer programming is in \mathcal{NP} is given. The proof also establishes that there is a pseudopolynomial-time algorithm for integer programming with any (fixed) number of constraints.

KEY WORDS AND PHRASES: integer linear programming, \mathcal{P} , \mathcal{NP} , pseudopolynomial algorithms

CR CATEGORIES: 5.25, 5.3, 5.4

So far...

- To establish NP-completeness of problem B, we always reduced Exact 3SAT to B
 - But we can reduce any other problem A that we have already established to be NP-complete
 - Sometimes this might lead to a simpler reduction because A might already be “similar” to B
- Let’s see an example!

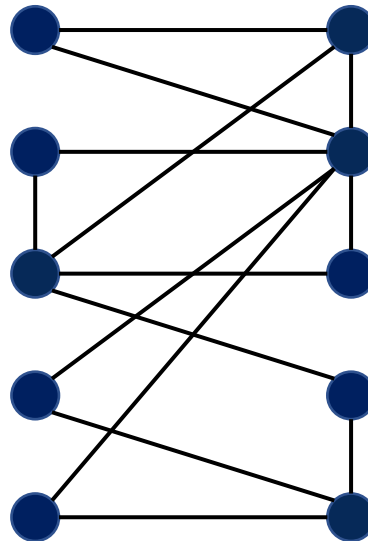
Vertex Cover

- **Problem**

- **Input:** Undirected graph $G = (V, E)$, integer k
- **Question:** Does there exist a vertex cover of size k ?
 - That is, does there exist $S \subseteq V$ with $|S| = k$ such that every edge is incident to at least one vertex in S ?

Example:

- Does this graph have a vertex cover of size 4?
 - Yes!
- Does this graph have a vertex cover of size 3?
 - No!



● = vertex cover

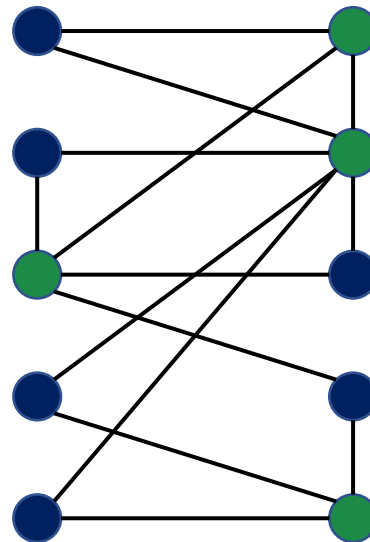
Vertex Cover

- **Problem**

- **Input:** Undirected graph $G = (V, E)$, integer k
- **Question:** Does there exist a vertex cover of size k ?
 - That is, does there exist $S \subseteq V$ with $|S| = k$ such that every edge is incident to at least one vertex in S ?

Question:

- Did we see this graph in the last lecture?
 - Yes!
 - For independent set of size 6



- = vertex cover
- = independent set

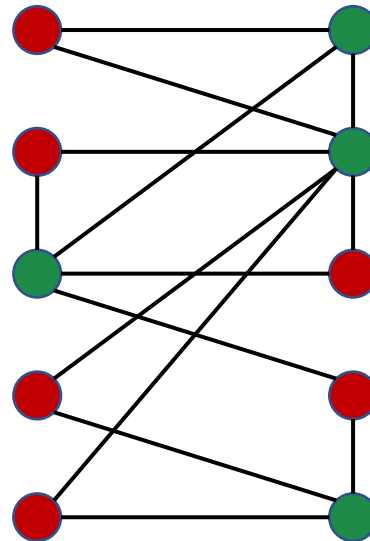
Vertex Cover

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 - That is, does there exist $S \subseteq V$ with $|S| = k$ such that every edge is incident to at least one vertex in S ?

Question:

- Did we see this graph in the last lecture?
 - Yes!
 - For independent set of size 6



- = vertex cover
- = independent set

Vertex Cover

- Vertex cover and independent set are intimately connected!
- **Claim:** G has a vertex cover of size k if and only if G has an independent set of size $n - k$
- **Stronger claim:** S is a vertex cover if and only if $V \setminus S$ is an independent set

Vertex Cover

- **Claim:** S is a vertex cover if and only if $V \setminus S$ is an independent set
- **Proof:**
 - S is a vertex cover
 - IFF: For every $(u, v) \in E$, at least one of $\{u, v\}$ is in S
 - IFF: For every $(u, v) \in E$, at most one of $\{u, v\}$ is in $V \setminus S$
 - IFF: No two vertices of $V \setminus S$ are connected by an edge
 - IFF: $V \setminus S$ is an independent set ■

Vertex Cover

- Claim: Independent Set \leq_p Vertex Cover
 - Take an arbitrary instance (G, k) of Independent Set
 - We want to check if there is an independent set of size k
 - Just convert it to the instance $(G, n - k)$ of Vertex Cover
 - Simple!
 - A reduction from Exact 3SAT would have basically repeated the reduction we already did for Exact 3SAT \leq_p Independent Set
 - **Note:** I didn't argue that Vertex Cover is in NP
 - This is simple as usual. Just give the actual vertex cover as the advice.

Set Cover

- **Problem**

- **Input:** A universe of elements U , a family of subsets S , and an integer k
- **Question:** Do there exist k sets from S whose union is U ?

- **Example**

- $U = \{1,2,3,4,5,6,7\}$
- $S = \{\{1,3,7\}, \{2,4,6\}, \{4,5\}, \{1\}, \{1,2,6\}\}$
- $k = 3$? Yes! $\{\{1,3,7\}, \{4,5\}, \{1,2,6\}\}$
- $k = 2$? No!

Set Cover

- Claim: Set Cover is in NP

- Easy. Let the advice be the actual k sets whose union is U .

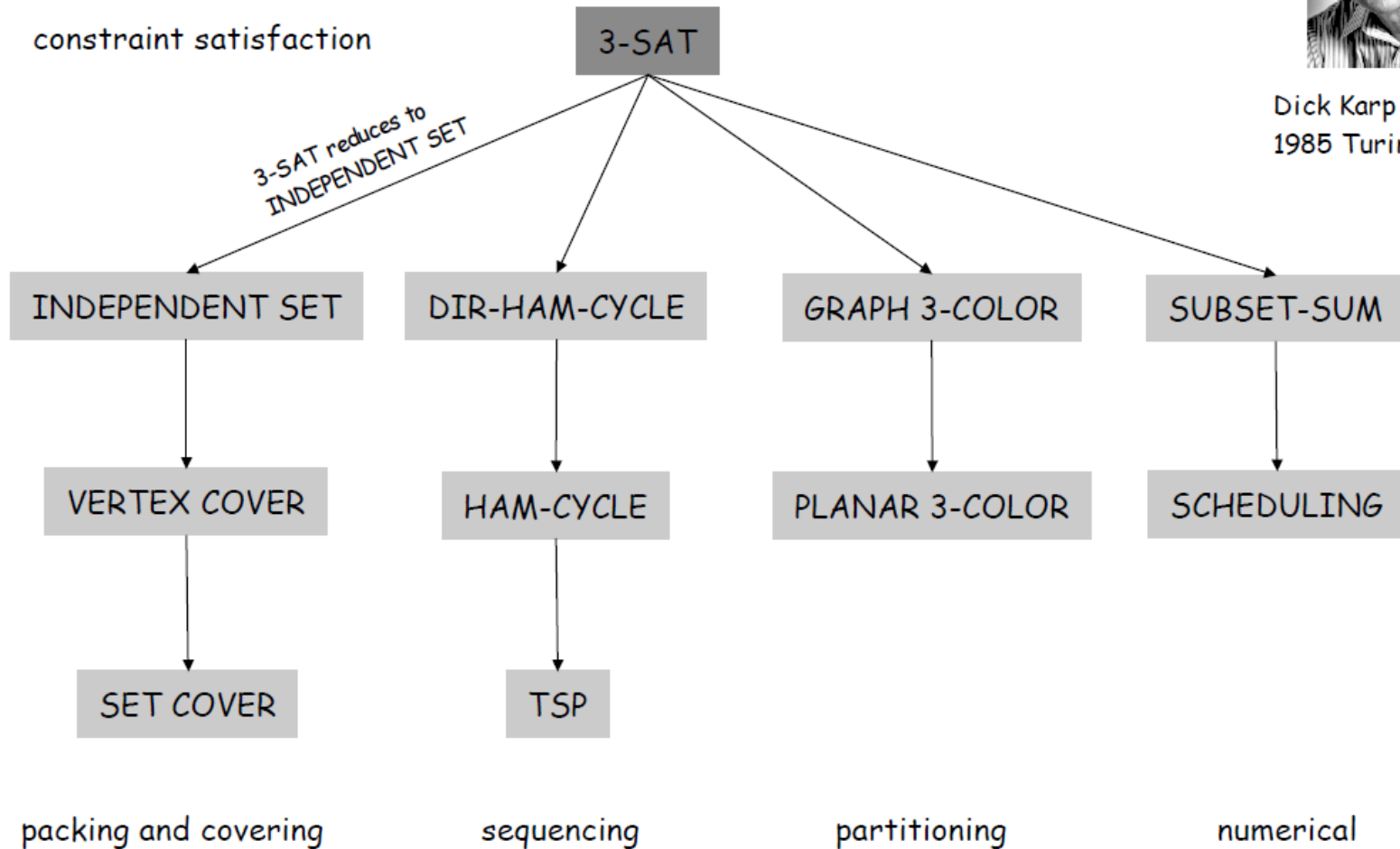
- Claim: Vertex Cover \leq_p Set Cover

- Given an instance of vertex cover with graph $G = (V, E)$ and integer k , create the following set cover instance
 - Set $U = E$
 - For each $v \in V$, S contains a set S_v of all the edges incident on v
 - Selecting k set whose union is $U =$ selecting k vertices such that union of their incident edges covers all edges
 - Hence, the two problems obviously have the same answer

Polynomial-Time Reductions



Dick Karp (1972)
1985 Turing Award



Cook-Levin Theorem

- We did not prove “the first NP-completeness” result
- **Theorem: Exact 3SAT is NP-complete**
 - We need to prove this without using any other “known NP-complete” problem
 - We want to directly show that *every problem in NP* can be reduced to Exact 3SAT
- We will first reduce any NP problem to SAT, and then reduce SAT to Exact 3SAT

Cook-Levin Theorem

- We're not going to prove it in this class, but the key idea is as follows
 - If a problem is in NP, then \exists Turing machine $T(x, y)$ which
 - takes as input a problem instance x and an advice y of size $p(|x|)$
 - verifies in $q(|x|)$ time whether x is a YES instance
 - both p and q are polynomials
 - x is a YES instance iff $\exists y T(x, y) = ACCEPT$

Cook-Levin Theorem

NOT IN SYLLABUS

- x is a YES instance iff $\exists y T(x, y) = ACCEPT$
 - We need to convert $\exists y T(x, y) = ACCEPT$ into whether a SAT formula φ is satisfiable
- Recall that a Turing machine T consists of a memory tape, a head pointer, a state, and a transition function
- **What describes T at any given step of its computation?**
 - What is written in each cell of its memory tape?
 - Which cell of the tape is the read/write head currently pointing to?
 - What state is the Turing machine in?

Cook-Levin Theorem

NOT IN SYLLABUS

- x is a YES instance iff $\exists y T(x, y) = ACCEPT$
 - We need to convert $\exists y T(x, y) = ACCEPT$ into $\exists z \varphi(z) = TRUE$, where z consists of Boolean variables and φ is a SAT formula
- **Variables:**
 - $T_{i,j,k}$ = True if machine's tape cell i contains symbol j at step k of the computation
 - $H_{i,k}$ = True if the machine's read/write head is at tape cell i at step k of the computation
 - $Q_{q,k}$ = True if machine is in state q at step k of the computation
 - Cell index i and computation step k only need to be polynomially large as T works in polynomial time

Cook-Levin Theorem

NOT IN SYLLABUS

- x is a YES instance iff $\exists y T(x, y) = ACCEPT$
 - We need to convert $\exists y T(x, y) = ACCEPT$ into $\exists z \varphi(z) = TRUE$, where z consists of Boolean variables and φ is a SAT formula
- **Clauses:**
 - Express how the variables must be related using the transition function
 - Express that the Turing machine must reach the state *ACCEPT* at some step of the computation
- This establishes that SAT is NP-complete.
- Next: $SAT \leq_p \text{Exact 3SAT}$.

Cook-Levin Theorem

- **Claim: $SAT \leq_p \text{Exact 3SAT}$**

- Take an instance $\varphi = C_1 \wedge C_2 \wedge \dots$ of SAT
- Replace each clause with multiple clauses with exactly 3 literals each

- **For a clause with one literal, $C = \ell_1$:**

- Add two variables z_1, z_2 , and replace C with four clauses

$$(\ell_1 \vee z_1 \vee z_2) \wedge (\ell_1 \vee \bar{z}_1 \vee z_2) \wedge (\ell_1 \vee z_1 \vee \bar{z}_2) \wedge (\ell_1 \vee \bar{z}_1 \vee \bar{z}_2)$$

- Verify that this is logically equivalent to ℓ_1

- **For a clause with two literals, $C = (\ell_1 \vee \ell_2)$:**

- Add variable z_1 and replace it with the following:

$$(\ell_1 \vee \ell_2 \vee z_1) \wedge (\ell_1 \vee \ell_2 \vee \bar{z}_1)$$

- Verify that this is logically equal to $(\ell_1 \vee \ell_2)$

Cook-Levin Theorem

- Claim: $SAT \leq_p \text{Exact 3SAT}$

- For a clause with three literals, $C = \ell_1 \vee \ell_2 \vee \ell_3$:

- Perfect. No need to do anything!

- For a clause with 4 or more literals, $C = (\ell_1 \vee \ell_2 \vee \dots \vee \ell_k)$:

- Add variables z_1, z_2, \dots, z_{k-3} and replace it with:

$$(\ell_1 \vee \ell_2 \vee z_1) \wedge (\ell_3 \vee \bar{z}_1 \vee z_2) \wedge (\ell_4 \vee \bar{z}_2 \vee z_3) \wedge \dots \\ \wedge (\ell_{k-2} \vee \bar{z}_{k-4} \vee z_{k-3}) \wedge (\ell_{k-1} \vee \ell_k \vee \bar{z}_{k-3})$$

- Check:

- If any ℓ_i is TRUE, then there exists an assignment of z variables to make this TRUE
- If all ℓ_i are FALSE, then no assignment of z variables will make this TRUE

NP vs co-NP

- **Complements of each other**

- NP = short proof for YES, co-NP = short proof for NO
- If a problem “Does there exist...” is in NP, then its complement “Does there not exist...” is in co-NP, and vice-versa
- The same goes for NP-complete and co-NP-complete

- **Example**

- SAT is NP-complete (“Does there exist x satisfying φ ?”)
 - So “Does there exist no x satisfying φ ?”, i.e., “Is φ always FALSE?” is coNP-complete
- Then, Tautology (“Is φ always TRUE?”) is also coNP-complete

$NP \cap co-NP$

- Clearly, $P \subseteq NP \cap co-NP$
 - No advice needed; can just solve the problem in polytime
 - Major open question: Is $P = NP \cap co-NP$?
- $NP \cap co-NP$: Short proof of both YES and NO
 - Hunt for problems not known in P but still in $NP \cap co-NP$

NP \cap co-NP

- Linear programming

- [Gale–Kuhn–Tucker 1948]: LP is in NP \cap co-NP
- **Question:** max objective value \geq threshold?
- **Proof of YES:** Provide a feasible solution with objective \geq threshold
- **Proof of NO:** Provide optimal primal and dual solutions

CHAPTER XIX

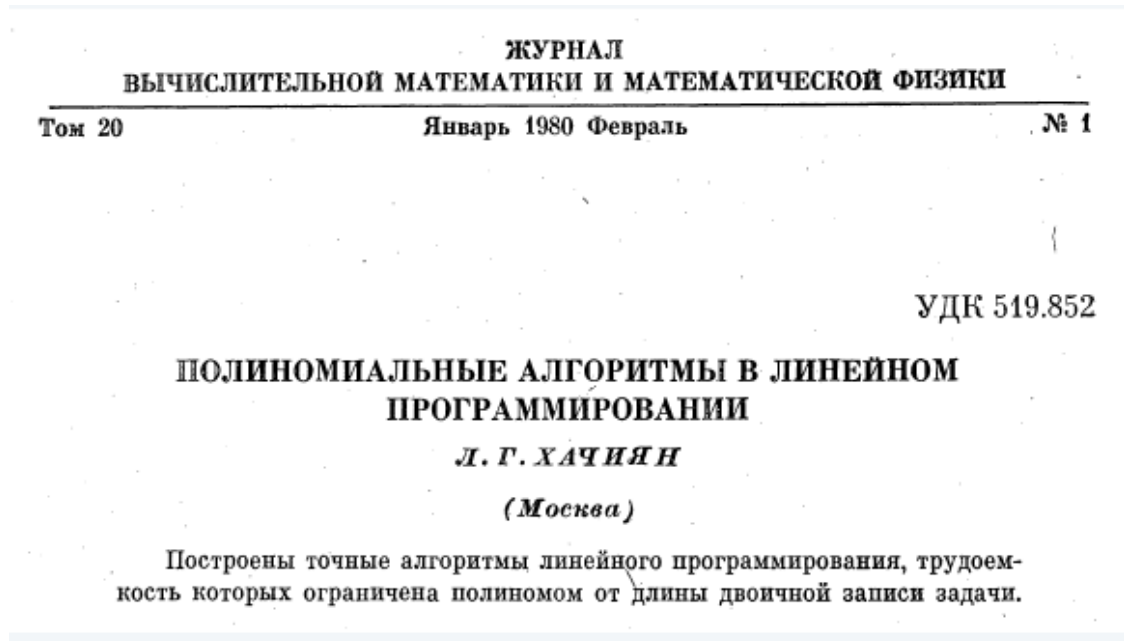
LINEAR PROGRAMMING AND THE THEORY OF GAMES¹

BY DAVID GALE, HAROLD W. KUHN, AND ALBERT W. TUCKER²

The basic “scalar” problem of *linear programming* is to maximize (or minimize) a linear function of several variables constrained by a system of linear inequalities [Dantzig, II]. A more general “vector” problem calls for maximizing (in a sense of partial order) a system of linear functions of several variables subject to a system of linear inequalities and, perhaps, linear equations [Koopmans, III]. The purpose of this chapter is to establish theorems of duality and existence for general “matrix” problems of linear programming which contain the “scalar” and “vector” problems as special cases, and to relate these general problems to the theory of zero-sum two-person games.

$NP \cap co-NP$

- Linear programming
 - But later, Khachiyan [1979] proved that LP is in P



NP \cap co-NP

- Primality testing (“Is n a prime?”)
 - [Pratt 1975]: PRIMES is in NP \cap co-NP
 - **Proof of NO:** Easy, provide a non-trivial factor
 - **Proof of YES:** relies on interesting math

SIAM J. COMPUT.
Vol. 4, No. 3, September 1975

EVERY PRIME HAS A SUCCINCT CERTIFICATE*

VAUGHAN R. PRATT†

Abstract. To prove that a number n is composite, it suffices to exhibit the working for the multiplication of a pair of factors. This working, represented as a string, is of length bounded by a polynomial in $\log_2 n$. We show that the same property holds for the primes. It is noteworthy that almost no other set is known to have the property that short proofs for membership or nonmembership exist for all candidates without being known to have the property that such proofs are easy to come by. It remains an open problem whether a prime n can be recognized in only $\log_2^\alpha n$ operations of a Turing machine for any fixed α .

The proof system used for certifying primes is as follows.

AXIOM. $(x, y, 1)$.

INFERENCE RULES.

R_1 : $(p, x, a), q \vdash (p, x, qa)$ provided $x^{(p-1)/q} \not\equiv 1 \pmod{p}$ and $q|(p-1)$.

R_2 : $(p, x, p-1) \vdash p$ provided $x^{p-1} \equiv 1 \pmod{p}$.

THEOREM 1. p is a theorem $\equiv p$ is a prime.

THEOREM 2. p is a theorem $\supset p$ has a proof of $[4 \log_2 p]$ lines.

NP \cap co-NP

- Primality testing (“Is n a prime?”)
 - Later, Agrawal, Kayal, and Saxena [2004] proved that PRIMES is in P
 - Milestone result!

Annals of Mathematics, 160 (2004), 781–793

PRIMES is in P

By MANINDRA AGRAWAL, NEERAJ KAYAL, and NITIN SAXENA*

Abstract

We present an unconditional deterministic polynomial-time algorithm that determines whether an input number is prime or composite.

NP \cap co-NP

- Factoring (“Does n have a factor $\leq k$?”)

- FACTOR is in NP \cap co-NP

- Proof of YES: Just present such a factor

- Proof of NO:

- Present the entire prime factorization of n along with a short proof that each presented factor is a prime
- Verifier TM can check that each factor is indeed a prime, their product is indeed n , and none of the factors is $\leq k$
 - Actually, proofs of primality are not required anymore since we know the TM can just run the AKS algorithm to check if the factors are prime

NP \cap co-NP

- Factoring (“Does n have a factor $\leq k$?”)
 - Major open question: Is FACTOR in P?
 - Basis of several cryptographic procedures
 - **Challenge:** Factor the following number.

74037563479561712828046796097429573142593188889231289
08493623263897276503402826627689199641962511784399589
43305021275853701189680982867331732731089309005525051
16877063299072396380786710086096962537934650563796359

RSA-704

(A \$30,000 prize was claimed in 2012 for this)

NP \cap co-NP

- Factoring (“Does n have a factor $\leq k$?”)
 - [Shor 1994]: We can factor an n -bit integer in $O(n^3)$ steps on a quantum computer.
 - *Scalable* quantum computers can help
 - 2001: Factored $15 = 3 \times 5$
 - 2012: Factored $21 = 3 \times 7$

Types of Reductions

- $A \leq_p B$
 - Karp reductions
 - Take an arbitrary instance of A , and in polynomial time, **construct a single instance of B with the same answer**
 - Very restricted type of reduction
 - The reduction we just constructed was a Karp reduction
 - Turing/Cook reductions
 - Take an arbitrary instance of A , and solve it by **making polynomially many calls to an oracle for solving B and some polynomial-time extra computation**
 - Very general reduction (B doesn't even have to be a decision problem)
 - In this course, we'll use Karp reductions only

Complexity Classes

- **Based on the exact time complexity**
 - $\text{DTIME}(n)$, $\text{NTIME}(n^2)$, ...
 - Deterministic / nondeterministic time complexity
- **Based on space complexity**
 - $\text{DSPACE}(n)$, $\text{NSPACE}(\log n)$
- **Using randomization**
 - ZPP (expected polynomial time, no errors)
 - Is $P = ZPP$?
- **Allowing probabilistic errors**
 - RP (polynomial time, one-sided error)
 - BPP (polynomial time, two-sided errors)