

CSC304 Fall'26
Solutions to Tutorial 5

Q1 Variants of the Vickrey Auction

Recall the single-item Vickrey auction from class. There are n agents, and each agent i has a private value $v_i \geq 0$ for the item. Given the reported values $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_n)$, the item is given to an agent $i^* \in \arg \max_i \tilde{v}_i$, who pays $\max_{j \neq i^*} \tilde{v}_j$, and the other agents pay nothing. In class, we saw that the Vickrey auction is strategyproof. In this question, we consider two variants of the Vickrey auction. As in the Vickrey auction, it does not matter how ties are broken between agents with tied maximum reported value, but for simplicity, you can assume that in such a case the item will be given to the agent with the lowest index (i.e., agent 1 before agent 2 before agent 3, and so on).

(a) The *third-price auction* (with $n \geq 3$ agents) gives the item to an agent $i^* \in \arg \max_i \tilde{v}_i$ as before, but charges her the third highest reported value; the other agents pay nothing. Is the third-price auction strategyproof? Prove it or provide a counterexample.

(b) The *Vickrey auction with reserve price* $r > 0$ gives the item to an agent $i^* \in \arg \max_i \tilde{v}_i$ only if $\tilde{v}_{i^*} \geq r$, in which case it charges her $\max(r, \max_{j \neq i^*} \tilde{v}_j)$. If $\tilde{v}_{i^*} < r$, the item is not sold. In either case, the other agents pay nothing. Is the Vickrey auction with reserve price r strategyproof? Is it welfare maximizing?

[Hint: For strategyproofness, imagine a fake agent $n + 1$ who always reports value r (note that he loses all ties), and interpret agent $n + 1$ winning the item as the item not being sold. Then, use the fact that the Vickrey auction is strategyproof, which we saw in class.]

Solution to Q1

(a) No, the third-price auction is not strategyproof. Consider $n = 3$ agents, and suppose agent 1 has true value $v_1 = 8$, while agents 2 and 3 report $\tilde{v}_2 = 10$ and $\tilde{v}_3 = 2$. If agent 1 reports her true value, agent 2 wins the item, and agent 1 has utility 0. If agent 1 instead reports $\tilde{v}_1 = 11$, she wins the item and pays the third highest reported value, which is 2, so her utility is $8 - 2 = 6 > 0$. Hence, agent 1 is strictly better off misreporting.

PS: Notice the contrast with the first-price auction from class (Rule 2), in which an agent has an incentive to report a value *lower* than her true value. In the third-price auction, an agent has an incentive to report a value *higher* than her true value: by doing so, she can win the item while paying only the second highest report among the other agents. The Vickrey auction charges the winner precisely the highest report among the other agents, i.e., the lowest report with which she would still win, which is what makes it strategyproof.

(b) The Vickrey auction with reserve price r is strategyproof, but not welfare maximizing.

Strategyproofness: Fix an agent $i \in \{1, \dots, n\}$, her true value v_i , and the reported values $\tilde{v}_{-i}$ of the other agents. Consider the Vickrey auction with $n + 1$ agents $1, \dots, n + 1$, in which agent $n + 1$ reports value r , and ties are broken in favor of the agent with the lowest index (so agent $n + 1$ loses

all ties). We claim that for every report $\tilde{v}_i$ of agent i , this Vickrey auction gives the item to the same agent and charges her the same amount as the Vickrey auction with reserve price r (where agent $n + 1$ receiving the item corresponds to the item not being sold). Let i^* denote the agent with the highest report among agents $1, \dots, n$ (breaking ties in favor of the lowest index).

- If $\tilde{v}_{i^*} \geq r$, then because agent $n + 1$ loses all ties, the Vickrey auction gives the item to agent i^* and charges her the highest report among the other agents, which is $\max(r, \max_{j \neq i^*} \tilde{v}_j)$, where the inner maximum is over agents $j \in \{1, \dots, n\} \setminus \{i^*\}$. This is precisely what the Vickrey auction with reserve price r does.
- If $\tilde{v}_{i^*} < r$, then agent $n + 1$ has the strictly highest report, so the Vickrey auction gives the item to agent $n + 1$, and agents $1, \dots, n$ receive nothing and pay nothing. This corresponds to the item not being sold, which is precisely what the Vickrey auction with reserve price r does.

Hence, for every report $\tilde{v}_i$, agent i has the same utility under both auctions. Because the Vickrey auction is strategyproof, reporting $\tilde{v}_i = v_i$ maximizes the utility of agent i under the Vickrey auction with agents $1, \dots, n + 1$ (this holds for all reports of the other agents, including agent $(n + 1)$'s report r). Hence, reporting $\tilde{v}_i = v_i$ also maximizes the utility of agent i under the Vickrey auction with reserve price r . Since this holds for every agent i , every true value v_i , and all reported values $\tilde{v}_{-i}$ of the other agents, the Vickrey auction with reserve price r is strategyproof.

Welfare maximization: Here, the set of outcomes consists of giving the item to agent i for each $i \in \{1, \dots, n\}$, and not selling the item. The welfare of giving the item to agent i is $\tilde{v}_i$, and the welfare of not selling the item is 0. Consider $n = 2$ agents who report $\tilde{v}_1 = 5$ and $\tilde{v}_2 = 3$, and a reserve price of $r = 10$. The item is not sold, which yields welfare 0, whereas giving the item to agent 1 yields welfare 5. Hence, the Vickrey auction with reserve price r is not welfare maximizing.

PS: Why would anyone use a reserve price then? Because it can increase the revenue of the auctioneer. For example, with two agents who report $\tilde{v}_1 = 10$ and $\tilde{v}_2 = 2$, the Vickrey auction earns revenue 2, whereas the Vickrey auction with reserve price $r = 8$ earns revenue 8.

Q2 Procurement Auction

A city wants to hire one of $n \geq 2$ contractors to build a bridge. Each contractor i has a private cost $c_i \geq 0$ for building the bridge, and incurs no cost if she is not hired. The city asks each contractor to report her cost, and given the reported costs $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_n)$, decides which contractor to hire and how much to pay each contractor.

This fits into the framework we saw in class. The set of outcomes A consists of n outcomes, one for hiring each contractor. The valuation of contractor i is $v_i(a) = -c_i$ for the outcome a in which she is hired, and $v_i(a) = 0$ for every other outcome a . If the city pays an amount x to contractor i , then $p_i = -x$, i.e., contractor i pays $-x$ to the city.

(a) Design a mechanism that always hires a contractor with the lowest reported cost (note that this is welfare maximizing), and is strategyproof and individually rational. Prove that your mechanism satisfies both properties.

(b) Does your mechanism satisfy no payments to agents? Can any mechanism that always hires a contractor satisfy both individual rationality and no payments to agents?

Solution to Q2

(a) Consider the following mechanism, which mirrors the Vickrey auction: hire a contractor $i^* \in \arg \min_i \tilde{c}_i$ (breaking ties in favor of the lowest index), pay her the second lowest reported cost $\min_{j \neq i^*} \tilde{c}_j$, and pay nothing to the other contractors. That is, $p_{i^*} = -\min_{j \neq i^*} \tilde{c}_j$, and $p_j = 0$ for all $j \neq i^*$.

Fix a contractor i , her true cost c_i , and the reported costs $\tilde{c}_{-i}$ of the other contractors. Let $b = \min_{j \neq i} \tilde{c}_j$ denote the lowest reported cost among the other contractors. If contractor i is hired, she is paid b , so her utility is $-c_i - (-b) = b - c_i$. If she is not hired, her utility is 0. Note that contractor i 's report only affects whether she is hired, and not the amount b she is paid if hired. Also, she can only be hired if $\tilde{c}_i \leq b$.

- Case 1: $c_i < b$. If contractor i reports truthfully, she has the strictly lowest reported cost, so she is hired and her utility is $b - c_i > 0$. Since every possible outcome gives her utility either $b - c_i$ or 0, she cannot do better by misreporting.
- Case 2: $c_i > b$. If contractor i reports truthfully, she is not hired (some other contractor reports $b < \tilde{c}_i$), so her utility is 0. Any report with which she is hired gives her utility $b - c_i < 0$. Hence, she cannot do better by misreporting.
- Case 3: $c_i = b$. Every possible outcome gives contractor i utility 0 (either $b - c_i = 0$ or 0), so she cannot do better by misreporting.

Hence, the mechanism is strategyproof. Further, in every case, contractor i has nonnegative utility when she reports truthfully, so the mechanism is individually rational.

PS: In fact, this mechanism is exactly the Vickrey auction applied to the values $v_i = -c_i$: the contractor with the highest value (i.e., the lowest cost) wins, and she “pays” the highest value

among the other contractors, i.e., $p_{i^*} = \max_{j \neq i^*} (-\tilde{c}_j) = -\min_{j \neq i^*} \tilde{c}_j$. The argument from class for the strategyproofness of the Vickrey auction never used the fact that values are nonnegative, so it directly applies here.

(b) No. The hired contractor i^* has $p_{i^*} = -\min_{j \neq i^*} \tilde{c}_j$, which is negative whenever the second lowest reported cost is positive.

In fact, no mechanism that always hires a contractor can satisfy both individual rationality and no payments to agents. To see this, suppose every contractor has true cost 1 and reports truthfully. The mechanism hires some contractor i . By individual rationality, her utility $-1 - p_i$ must be nonnegative, so $p_i \leq -1 < 0$, which violates no payments to agents. This is an example of the settings mentioned in class in which we want payments in the reverse direction, i.e., from the center to the agents.