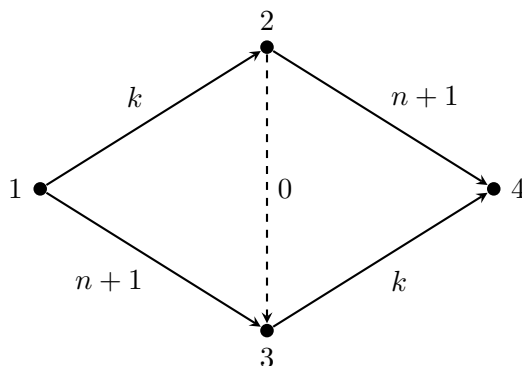


CSC304 Fall'26
Solutions to Tutorial 4

Q1 Braess' Paradox

Consider the following variant of the Braess' paradox example from class. There are n players, where $n \geq 4$ is even, and each player wants to travel from node 1 to node 4 in the road network shown below. Roads $1 \rightarrow 2$ and $3 \rightarrow 4$ are congestible: if k players take such a road, each of them incurs a cost of k on that road. Roads $1 \rightarrow 3$ and $2 \rightarrow 4$ are not congestible: each player taking such a road incurs a cost of $n + 1$ on that road, regardless of how many players take it. The cost to a player is the total cost she incurs on the roads on her path.



- (a) First, suppose that the (dashed) road $2 \rightarrow 3$ does not exist. Find all pure Nash equilibria of this game and the cost to each player under them.
- (b) Now, suppose that the road $2 \rightarrow 3$, with a cost of 0 to each player taking it, is added to the network. Show that the resulting game has a unique Nash equilibrium, and find the cost to each player under it.
- (c) By what factor does adding the road $2 \rightarrow 3$ increase the cost to each player in equilibrium? What does this factor converge to as $n \rightarrow \infty$?
- (d) For the network with the road $2 \rightarrow 3$, find the optimal social cost (i.e., the minimum total cost to all players) and the price of anarchy.

Solution to Q1

(a) Without the road $2 \rightarrow 3$, each player takes either the upper path $1 \rightarrow 2 \rightarrow 4$ or the lower path $1 \rightarrow 3 \rightarrow 4$. Suppose k players take the upper path and $n - k$ players take the lower path. Then, each player taking the upper path has cost $k + (n + 1)$, and each player taking the lower path has cost $(n + 1) + (n - k)$.

- If $k > n/2$, then $k \geq n/2 + 1$ because n is even. A player taking the upper path would have cost $(n + 1) + (n - k + 1)$ by switching to the lower path, which is strictly less than her current cost $k + (n + 1)$ because $n - k + 1 < k$. Hence, this is not a pure Nash equilibrium.

- If $k < n/2$, then by a symmetric argument, a player taking the lower path would strictly decrease her cost by switching to the upper path.
- If $k = n/2$, then each player has cost $n/2 + n + 1 = 3n/2 + 1$. A player switching to the other path would have cost $(n + 1) + (n/2 + 1) > 3n/2 + 1$. Hence, this is a pure Nash equilibrium.

Thus, the pure Nash equilibria are precisely the strategy profiles in which $n/2$ players take each path, and the cost to each player under them is $3n/2 + 1$.

(b) With the road $2 \rightarrow 3$, each player can also take the zigzag path $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$. We show that, for each player, the zigzag path strictly dominates the other two paths. Fix a player i and the paths taken by the other players.

- Compare the zigzag path with the upper path $1 \rightarrow 2 \rightarrow 4$. Both paths take the road $1 \rightarrow 2$, which therefore has the same number of players under both choices of player i . Apart from that, the zigzag path takes the roads $2 \rightarrow 3$ and $3 \rightarrow 4$, with a total cost of at most $0 + n$ (at most n players take the road $3 \rightarrow 4$), while the upper path takes the road $2 \rightarrow 4$, with a cost of $n + 1$. Hence, the zigzag path is strictly cheaper for player i .
- Similarly, compared to the lower path $1 \rightarrow 3 \rightarrow 4$, the zigzag path shares the road $3 \rightarrow 4$ and replaces the road $1 \rightarrow 3$ (cost $n + 1$) with the roads $1 \rightarrow 2$ and $2 \rightarrow 3$ (total cost at most $n + 0$). Hence, the zigzag path is strictly cheaper for player i .

Because this holds for every choice of paths by the other players, it also holds in expectation when the other players randomize. Hence, in any Nash equilibrium, no player can assign a positive probability to the upper or the lower path, as moving this probability to the zigzag path would strictly decrease her expected cost. Further, all players taking the zigzag path is a Nash equilibrium because the zigzag path is a best response to any strategies of the other players. Thus, the unique Nash equilibrium is for every player to take the zigzag path. Each of the roads $1 \rightarrow 2$ and $3 \rightarrow 4$ is then taken by all n players, so the cost to each player is $n + 0 + n = 2n$.

(c) Adding the road $2 \rightarrow 3$ increases the cost to each player in equilibrium from $3n/2 + 1$ to $2n$, i.e., by a factor of

$$\frac{2n}{3n/2 + 1} = \frac{4n}{3n + 2},$$

which converges to $4/3$ as $n \rightarrow \infty$. For example, for $n = 4$, the factor is $8/7$, the same factor as in the example from class (where the cost increased from 35 to 40).

(d) Consider any pure strategy profile, and let x and y denote the numbers of players taking the roads $1 \rightarrow 2$ and $3 \rightarrow 4$, respectively. Each player on the zigzag path takes both of these roads, while every other player takes exactly one of them, so the number of players on the zigzag path is $x + y - n$, and the number of players taking a road with cost $n + 1$ (each such player takes exactly one) is $n - (x + y - n) = 2n - x - y$. The total cost to the players on a congestible road taken by k players is $k \cdot k = k^2$. Hence, the social cost is

$$x^2 + y^2 + (n + 1)(2n - x - y) = \left(x - \frac{n + 1}{2}\right)^2 + \left(y - \frac{n + 1}{2}\right)^2 + 2n(n + 1) - \frac{(n + 1)^2}{2}.$$

Because n is even, $(n+1)/2$ is not an integer, while x and y are integers. Hence, each of the two squared terms is at least $1/4$, and the social cost is at least

$$\frac{1}{2} + 2n(n+1) - \frac{(n+1)^2}{2} = \frac{3n^2}{2} + n.$$

This is achieved by the pure Nash equilibrium from part (a), in which $x = y = n/2$ and no player uses the road $2 \rightarrow 3$ (and which is still a valid strategy profile after adding the road). Hence, the optimal social cost is $n \cdot (3n/2 + 1)$.

The social cost in the unique Nash equilibrium from part (b) is $n \cdot 2n$. Hence, the price of anarchy (which here coincides with the price of stability) is

$$\frac{n \cdot 2n}{n \cdot (3n/2 + 1)} = \frac{4n}{3n + 2},$$

which is the same factor as in part (c). In other words, after the road $2 \rightarrow 3$ is added, the socially optimal strategy profile simply ignores it, but the players cannot resist using it.

Q2 Zero-Sum Games

Consider the following two-player zero-sum game, where the numbers indicate the reward to the row player.

Row \ Column	L	R
T	5	-3
B	-4	2

- (a) What is the maximin value and a maximin strategy for the row player?
- (b) What is the minimax value and a minimax strategy for the column player?

[Note: You do not need to solve a complex linear program for this. E.g., if you assume the row player's maximin strategy to be $p \cdot T + (1 - p) \cdot B$, you can compute the reward to the row player when the column player chooses L and when the column player chooses R . You want to find the value of p that maximizes the minimum of the two rewards. This can be found by plotting both rewards using any online graphing tool such as Desmos.]

Solution to Q2

(a) Suppose the row player plays the mixed strategy $p \cdot T + (1 - p) \cdot B$ for some $p \in [0, 1]$. The reward to the row player...

- When the column player chooses $L = 5p - 4(1 - p) = 9p - 4$
- When the column player chooses $R = -3p + 2(1 - p) = 2 - 5p$

The worst-case reward is $\min(9p - 4, 2 - 5p)$. As seen in class, the maximin strategy is the one that maximizes this worst-case reward.

We can find this by plotting both $9p - 4$ and $2 - 5p$ for $p \in [0, 1]$ using any online tool and choosing the value of p where the smallest of the two terms is maximized.

Or we can notice that $9p - 4$ is increasing in p while $2 - 5p$ is decreasing in p . In such cases, the smallest of the two terms is maximized when both terms are equal (if they're ever equal). Setting both terms equal, we get $p = 3/7$.

Hence, the maximin strategy of the row player is $(3/7) \cdot T + (4/7) \cdot B$. For the maximin value, we plug in this p into either reward expression to get $9(3/7) - 4 = 2 - 5(3/7) = -1/7$.

(b) From the minimax theorem, the minimax value of the column player is equal to the maximin value of the row player, i.e., $-1/7$. We can compute the column player's minimax strategy similarly to above.

Assuming the column player plays $q \cdot L + (1 - q) \cdot R$, the reward to the row player in the worst case for the column player would be $\max(5q - 3(1 - q), -4q + 2(1 - q)) = \max(8q - 3, 2 - 6q)$, and the minimax strategy minimizes this. Because $8q - 3$ is increasing in q while $2 - 6q$ is decreasing in q , this is minimized when $8q - 3 = 2 - 6q \Rightarrow q = 5/14$. Hence, the minimax strategy of the column player is $(5/14) \cdot L + (9/14) \cdot R$.

Q3 No, You Go First!

Consider the following normal-form representation of a game. The row player John and the column player Marina each have three actions: L , M , and R .

John \ Marina	L	M	R
L	$(0, 1)$	$(-2, -1)$	$(-\frac{3}{2}, \frac{2}{3})$
M	$(-1, -2)$	$(-1, 0)$	$(-3, -1)$
R	$(1, 0)$	$(-2, -1)$	$(-2, \frac{1}{2})$

- (a) What is the (unique) pure Nash equilibrium of this game? What are John and Marina's rewards in this case?
- (b) If John can move first and commit to a pure strategy, which pure strategy should he commit to, and what would be the outcome? Explain.
- (c) If Marina can move first and commit to a pure strategy, which pure strategy should she commit to, and what would be the outcome? Explain.
- (d) Compare the rewards of John and Marina across the three possibilities (simultaneous moves, John first, Marina first), and say something interesting.
- (e) Now, suppose that the player who moves first can commit to a *mixed* strategy. As in class, if the player who moves second has multiple best responses, assume that she breaks ties in favor of the player who moves first. If John moves first, which mixed strategy should he commit to, and what would be the outcome? Explain.
- (f) If Marina moves first and can commit to a mixed strategy, which mixed strategy should she commit to, and what would be the outcome? Explain.
- (g) Redo the comparison from part (d) with mixed commitments. What changes?

[Hint: As in class, for each action of the player who moves second, find the best mixed strategy for the player who moves first subject to this action being a best response of the player who moves second.]

Solution to Q3

- (a) The unique pure Equilibrium is (M, M) . In this case, John gets -1 and Marina gets 0 .
- (b) Here are the various situations when John commits first.

John commits	Marina responds	John's payoff	Marina's payoff
L	L	0	1
M	M	-1	0
R	R	-2	$1/2$

Hence, John should commit to L , which gives him reward 0 and Marina reward 1.

(c) Here are the various situations when Marina commits first.

Marina commits	John responds	John's payoff	Marina's payoff
L	R	1	0
M	M	-1	0
R	L	$-3/2$	$2/3$

Hence, Marina should commit to R , which gives John reward $-3/2$ and herself reward $2/3$.

(d) Here are the rewards to both players in all three cases.

Player	Simultaneous	John first	Marina first
John	-1	0	$-3/2$
Marina	0	1	$2/3$

For John, committing first is better than playing a simultaneous move game (committing first is always at least as good), which is in turn better than letting Marina commit first.

For Marina, committing first is better than playing a simultaneous move game (committing first is always at least as good), but interestingly, it's even better to let John commit first.

(e) Suppose John commits to playing L , M , and R with probabilities p_L , p_M , and p_R , respectively. Then, Marina's expected rewards for her three actions are $p_L - 2p_M$ for L , $-p_L - p_R$ for M , and $\frac{2}{3}p_L - p_M + \frac{1}{2}p_R$ for R . As in class, we consider each possible response of Marina separately.

- If Marina responds with M , John's expected reward is $-2p_L - p_M - 2p_R = -1 - p_L - p_R \leq -1$.
- If Marina responds with R , John's expected reward is $-\frac{3}{2}p_L - 3p_M - 2p_R \leq -\frac{3}{2}$.
- If Marina responds with L , John's expected reward is $p_R - p_M$. For L to be a best response of Marina, it must yield at least as much reward as R , i.e., $p_L - 2p_M \geq \frac{2}{3}p_L - p_M + \frac{1}{2}p_R$, which simplifies to $2p_L \geq 6p_M + 3p_R$. Since $p_M \geq 0$ and $p_L \leq 1 - p_R$, this implies $2(1 - p_R) \geq 3p_R$, i.e., $p_R \leq 2/5$. Hence, John's expected reward is $p_R - p_M \leq 2/5$.

The bound of $2/5$ is achieved when John commits to $(p_L, p_M, p_R) = (3/5, 0, 2/5)$. Under this strategy, Marina's expected rewards are $3/5$ for L , -1 for M , and $3/5$ for R . So, both L and R are best responses, and Marina breaks the tie in favor of John by choosing L (which gives John $2/5$, whereas R would give him $-17/10$). Since $2/5$ is higher than what John can get when Marina responds with M or R , John should commit to $(3/5, 0, 2/5)$. In the resulting outcome, Marina responds with L , John gets $2/5$, and Marina gets $3/5$.

(f) Suppose Marina commits to playing L , M , and R with probabilities q_L , q_M , and q_R , respectively. Then, John's expected rewards for his three actions are $-2q_M - \frac{3}{2}q_R$ for L , $-q_L - q_M - 3q_R$ for M , and $q_L - 2q_M - 2q_R$ for R . Again, we consider each possible response of John separately.

- If John responds with M , Marina's expected reward is $-2q_L - q_R \leq 0$.

- If John responds with R , Marina's expected reward is $-q_M + \frac{1}{2}q_R$. For R to be a best response of John, it must yield at least as much reward as L , i.e., $q_L - 2q_M - 2q_R \geq -2q_M - \frac{3}{2}q_R$, which simplifies to $q_L \geq q_R/2$. Since $q_L + q_R \leq 1$, this implies $q_R \leq 2/3$. Hence, Marina's expected reward is at most $\frac{1}{2}q_R \leq 1/3$.
- If John responds with L , Marina's expected reward is $q_L - q_M + \frac{2}{3}q_R$. For L to be a best response of John, it must yield at least as much reward as R , i.e., $-2q_M - \frac{3}{2}q_R \geq q_L - 2q_M - 2q_R$, which simplifies to $q_R \geq 2q_L$. Since $q_L + q_R \leq 1$, this implies $q_L \leq 1/3$. Hence, Marina's expected reward is $q_L - q_M + \frac{2}{3}q_R \leq q_L + \frac{2}{3}(1 - q_L) = \frac{2}{3} + \frac{1}{3}q_L \leq \frac{7}{9}$.

The bound of $7/9$ is achieved when Marina commits to $(q_L, q_M, q_R) = (1/3, 0, 2/3)$. Under this strategy, John's expected rewards are -1 for L , $-7/3$ for M , and -1 for R . So, both L and R are best responses, and John breaks the tie in favor of Marina by choosing L (which gives Marina $7/9$, whereas R would give her $1/3$). Since $7/9$ is higher than what Marina can get when John responds with M or R , Marina should commit to $(1/3, 0, 2/3)$. In the resulting outcome, John responds with L , John gets -1 , and Marina gets $7/9$.

(g) Here are the rewards to both players in all three cases, where the player moving first can now commit to a mixed strategy.

Player	Simultaneous	John first	Marina first
John	-1	$2/5$	-1
Marina	0	$3/5$	$7/9$

Being able to commit to a mixed strategy helps the player who moves first: John's reward when he moves first increases from 0 to $2/5$, and Marina's reward when she moves first increases from $2/3$ to $7/9$.

Interestingly, the observation from part (d) flips. With pure commitments, Marina preferred to let John commit first (1) over committing first herself ($2/3$). With mixed commitments, Marina prefers to commit first herself ($7/9$) over letting John commit first ($3/5$). Also, John is no longer worse off when Marina moves first than when both players move simultaneously.