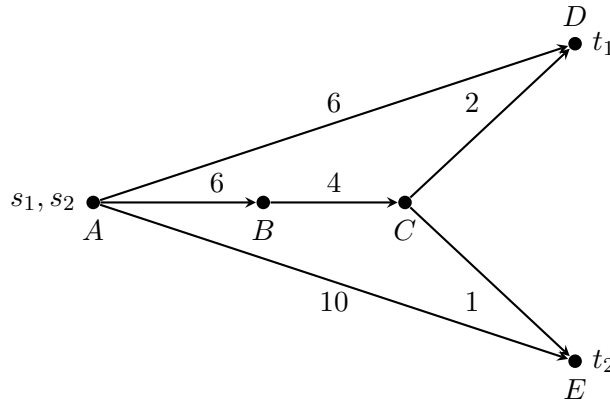


CSC304 Fall'26
Solutions to Tutorial 3

Q1 Cost-sharing games

Consider the cost-sharing game shown in the figure below. There are two players, one wishing to go from s_1 to t_1 and the other from s_2 to t_2 (both s_1 and s_2 are the same node A). The costs of the edges are indicated in the figure.



- (a) Find a socially optimal strategy profile. What is the optimal social cost? Is this strategy profile a Nash equilibrium?
- (b) Find all pure Nash equilibria of this game and social costs under these equilibria.
- (c) What are the prices of anarchy and stability (with respect to pure Nash equilibria) of this game?

Solution to Q1

(a) The social optimum is for P1 to choose $A \rightarrow B \rightarrow C \rightarrow D$ and for P2 to choose $A \rightarrow B \rightarrow C \rightarrow E$, which results in a social cost of $6 + 4 + 2 + 1 = 13$. This strategy profile is not a Nash equilibrium: P1 pays $6/2 + 4/2 + 2 = 7$ under it, but would pay only 6 by switching to $A \rightarrow D$.

(b) There are only four pure strategy profiles, so we can potentially enumerate over them. But we can also use the fact that iterated elimination of strictly dominated actions does not change the set of Nash equilibria of the game.

It is easy to see that for P1, $A \rightarrow D$ (cost at most 6) strictly dominates $A \rightarrow B \rightarrow C \rightarrow D$ (cost at least $6/2 + 4/2 + 2 = 7$). After eliminating $A \rightarrow B \rightarrow C \rightarrow D$ for P1, we find that for P2, $A \rightarrow E$ (cost 10) now strictly dominates $A \rightarrow B \rightarrow C \rightarrow E$ (cost $6 + 4 + 1 = 11$). Hence, we can eliminate $A \rightarrow B \rightarrow C \rightarrow E$ for P2 as well, resulting in the unique strategy profile (P1: $A \rightarrow D$, P2: $A \rightarrow E$), which must be a pure Nash equilibrium. The social cost under this equilibrium is $6 + 10 = 16$.

(c) Both the prices of anarchy and stability are $16/13$.

Q2 Price of stability in cost-sharing games

In class, we saw that the price of stability of any cost-sharing game is $O(\log n)$, where n is the number of players. Construct a cost-sharing game with n players in which the price of stability is $\Omega(\log n)$, thus establishing tightness of this bound.

Solution to Q2

Consider the cost-sharing game given in the figure below.

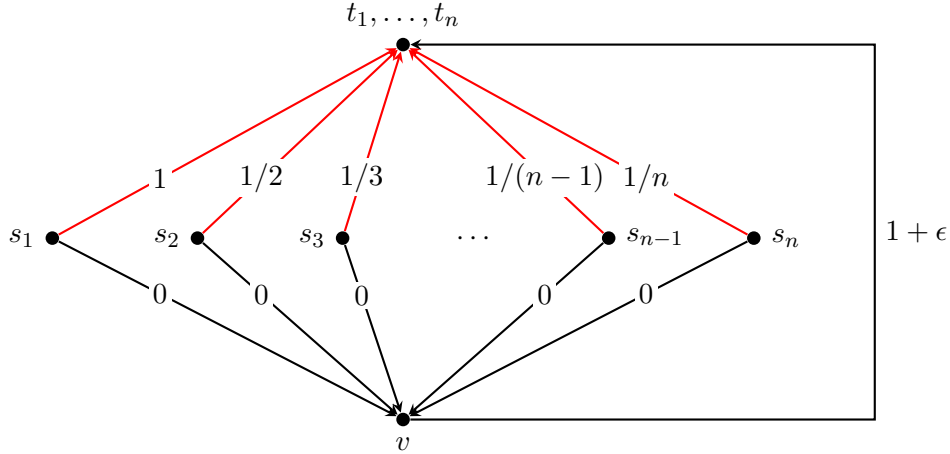


Figure 1: A cost-sharing game where the price of stability is $\Omega(\log n)$.

Each player $i \in \{1, \dots, n\}$ wants to go from node s_i to node t_i , where $t_1 = t_2 = \dots = t_n$ (let us call this node t). Each player i can take one of two paths: either the direct path consisting of an edge from s_i to t with cost $1/i$, or the non-direct path consisting of an edge from s_i to v with cost 0 followed by an edge from v to t with cost $1 + \epsilon$.

We claim that the only pure Nash equilibrium in this game is where each player i chooses her direct path. To see that this is a pure Nash equilibrium, note that if player i switches to her non-direct path, she becomes the only player taking the edge from v to t , so her cost increases from $1/i$ to $1 + \epsilon$. To see that there is no other pure Nash equilibrium, suppose for contradiction that there is a pure Nash equilibrium in which not all players choose their direct paths. Let k denote the player with the maximum index who does not choose the direct path. Then, at most k players choose the non-direct path. Hence, the cost to player k is at least $(1 + \epsilon)/k$. However, in this case, player k would be strictly incentivized to switch to her direct path, which only costs her $1/k$.

Hence, we have established that the only pure Nash equilibrium in the game is where each player chooses her direct path. This equilibrium has social cost $\sum_{i=1}^n 1/i = H(n)$. In contrast, all players choosing the non-direct path would have social cost $1 + \epsilon$. Hence, the price of stability is at least $H(n)/(1 + \epsilon)$. For constant $\epsilon > 0$, this is $\Omega(\log n)$.

Q3 Potential Functions

Recall that a potential function for a game assigns a real number $\Phi(\vec{s})$ to every pure strategy profile $\vec{s}$ such that whenever a single player changes her strategy, the change in the potential function is equal to the change in the cost to that player. That is, for every pure strategy profile $\vec{s}$, player i , and strategy s'_i of player i ,

$$c_i(s'_i, \vec{s}_{-i}) - c_i(\vec{s}) = \Phi(s'_i, \vec{s}_{-i}) - \Phi(\vec{s}).$$

In both games below, the first number in each cell is the cost to P1 and the second number is the cost to P2. Each player wants to minimize her cost.

(a) Construct a potential function for the following game by writing down the value of the potential function in each cell. Which cell minimizes your potential function? Is it a pure Nash equilibrium? Are there any other pure Nash equilibria of the game?

P1 \ P2	L	R
T	(3,5)	(7,8)
M	(5,6)	(3,3)
B	(7,6)	(5,3)

(b) Show that the following game does not have a potential function, even though it has a pure Nash equilibrium.

P1 \ P2	L	R
T	(2,2)	(3,1)
B	(1,4)	(2,2)

Solution to Q3

(a) Let $\Phi(x, y)$ denote the value of the potential function in the cell where P1 plays x and P2 plays y . Adding the same constant to the values in all cells does not affect the defining property of a potential function, so we can fix the value of one cell arbitrarily; let us set $\Phi(T, L) = 2$. The defining property then pins down the values of all other cells.

- When P2 switches from L to R in row T , her cost changes by $8 - 5 = 3$. Hence, $\Phi(T, R) = 2 + 3 = 5$.
- When P1 switches from T to M and from T to B in column L , her cost changes by $5 - 3 = 2$ and $7 - 3 = 4$, respectively. Hence, $\Phi(M, L) = 4$ and $\Phi(B, L) = 6$.
- When P2 switches from L to R in rows M and B , her cost changes by $3 - 6 = -3$ in both rows. Hence, $\Phi(M, R) = 4 - 3 = 1$ and $\Phi(B, R) = 6 - 3 = 3$.

This gives the following potential function.

P1 \ P2	L	R
T	2	5
M	4	1
B	6	3

We still need to check the deviations that we did not use above. In column L , when P1 switches from M to B , her cost changes by $7 - 5 = 2 = \Phi(B, L) - \Phi(M, L)$. In column R , when P1 switches from T to M , from M to B , and from T to B , her cost changes by $3 - 7 = -4$, $5 - 3 = 2$, and $5 - 7 = -2$, which are equal to $\Phi(M, R) - \Phi(T, R)$, $\Phi(B, R) - \Phi(M, R)$, and $\Phi(B, R) - \Phi(T, R)$, respectively. Deviations in the opposite directions simply flip the signs. Hence, Φ is a potential function. (Any other potential function for this game is obtained by adding a constant to the values in all cells. This is because, once we fixed the value of $\Phi(T, L)$, the defining property uniquely determined the values of all other cells.)

The cell minimizing the potential function is (M, R) , with potential value 1. As we saw in class, a pure strategy profile minimizing the potential function is a pure Nash equilibrium. Indeed, under (M, R) , P1 has cost 3 and would have cost 7 or 5 by switching to T or B , and P2 has cost 3 and would have cost 6 by switching to L .

The game has exactly one other pure Nash equilibrium, (T, L) . To see this, note that because the change in the cost to a player who changes her strategy is equal to the change in the potential function, a cell is a pure Nash equilibrium if and only if its potential value is the lowest in its column (so that P1 cannot decrease her cost) and the lowest in its row (so that P2 cannot decrease her cost). In the table above, this holds for (M, R) and (T, L) . It does not hold for (T, R) and (B, R) because (M, R) has a lower value in column R , and it does not hold for (M, L) and (B, L) because (M, R) and (B, R) have lower values in rows M and B , respectively. Indeed, under (T, L) , P1 has cost 3 and would have cost 5 or 7 by switching to M or B , and P2 has cost 5 and would have cost 8 by switching to R .

PS: Notice that (T, L) is a pure Nash equilibrium even though it does not minimize the potential function.

(b) First, (B, R) is a pure Nash equilibrium: B strictly dominates T for P1, and R strictly dominates L for P2.

Suppose for contradiction that Φ is a potential function for this game. Consider the cycle of strategy profiles $(T, L) \rightarrow (T, R) \rightarrow (B, R) \rightarrow (B, L) \rightarrow (T, L)$, in which P2, P1, P2, and P1 change their strategies in turn. By the definition of a potential function,

$$\begin{aligned}\Phi(T, R) - \Phi(T, L) &= c_2(T, R) - c_2(T, L) = 1 - 2 = -1, \\ \Phi(B, R) - \Phi(T, R) &= c_1(B, R) - c_1(T, R) = 2 - 3 = -1, \\ \Phi(B, L) - \Phi(B, R) &= c_2(B, L) - c_2(B, R) = 4 - 2 = 2, \\ \Phi(T, L) - \Phi(B, L) &= c_1(T, L) - c_1(B, L) = 2 - 1 = 1.\end{aligned}$$

Adding the four equations, the left-hand sides sum to 0 because the cycle returns to where it started, whereas the right-hand sides sum to $-1 - 1 + 2 + 1 = 1$. This is a contradiction. Hence, the game does not have a potential function.

PS: In class, we saw that having a potential function implies the existence of a pure Nash equilibrium. This example shows that the converse is not true.

Q4 Market Sharing Games

Consider the following game. There are n players and m markets. Each market $j \in \{1, \dots, m\}$ has a value $v_j > 0$. Each player chooses a single market to enter, and the value of each market is split equally among all players who enter it. Formally, given a pure strategy profile $\vec{s} = (s_1, \dots, s_n)$, where $s_i \in \{1, \dots, m\}$ is the market chosen by player i , let $n_j(\vec{s})$ denote the number of players choosing market j . Then, the reward to player i is $u_i(\vec{s}) = v_{s_i}/n_{s_i}(\vec{s})$. The social welfare of $\vec{s}$ is the total reward of the players, $W(\vec{s}) = \sum_{i=1}^n u_i(\vec{s})$. Throughout this question, the price of anarchy is with respect to pure Nash equilibria.

(a) A potential function for a game with rewards is defined analogously to that for a game with costs: for every pure strategy profile $\vec{s}$, player i , and strategy s'_i of player i , we need $u_i(s'_i, \vec{s}_{-i}) - u_i(\vec{s}) = \Phi(s'_i, \vec{s}_{-i}) - \Phi(\vec{s})$. Construct a potential function for this game, and conclude that the game always has a pure Nash equilibrium.

(b) Show that the price of anarchy of this game is at most 2.

[Hint: Fix a pure Nash equilibrium $\vec{s}$ and a socially optimal strategy profile $\vec{o}$. For each player i , use the fact that she cannot increase her reward by switching from s_i to o_i , and sum over all players. Then, consider the markets chosen by at least one player under $\vec{o}$ but by no player under $\vec{s}$.]

(c) Show that the bound in part (b) is almost tight: for every n , construct a game with n players whose price of anarchy is at least $2 - 1/n$.

Solution to Q4

Let $M(\vec{s})$ denote the set of markets chosen by at least one player under $\vec{s}$. Because the value of each market in $M(\vec{s})$ is split fully among the players choosing it, we have $W(\vec{s}) = \sum_{j \in M(\vec{s})} v_j$.

(a) Similarly to the potential function for cost-sharing games that we saw in class, define

$$\Phi(\vec{s}) = \sum_{j \in M(\vec{s})} \sum_{k=1}^{n_j(\vec{s})} \frac{v_j}{k}.$$

Suppose player i switches from market $j = s_i$ to market $j' \neq j$ (if $j' = j$, nothing changes). Her reward changes from $v_j/n_j(\vec{s})$ to $v_{j'}/(n_{j'}(\vec{s}) + 1)$. In the potential function, the number of players choosing market j decreases by one, so its inner sum loses its last term $v_j/n_j(\vec{s})$ (if $n_j(\vec{s}) = 1$, market j leaves $M(\vec{s})$ and its inner sum, which consisted of this single term, disappears). Similarly, the inner sum of market j' gains the term $v_{j'}/(n_{j'}(\vec{s}) + 1)$ (if $n_{j'}(\vec{s}) = 0$, market j' joins $M(\vec{s})$ with this single term). The inner sums of all other markets are unchanged. Hence, the change in the potential function is $v_{j'}/(n_{j'}(\vec{s}) + 1) - v_j/n_j(\vec{s})$, which is precisely the change in the reward to player i .

Because there are finitely many pure strategy profiles, there is a pure strategy profile $\vec{s}^*$ maximizing Φ . For every player i and strategy s'_i of player i , we have

$$u_i(s'_i, \vec{s}_{-i}^*) - u_i(\vec{s}^*) = \Phi(s'_i, \vec{s}_{-i}^*) - \Phi(\vec{s}^*) \leq 0.$$

Hence, no player can increase her reward by changing her strategy, i.e., $\vec{s}^*$ is a pure Nash equilibrium.

(b) Fix any pure Nash equilibrium $\vec{s}$ and a socially optimal strategy profile $\vec{o}$. We show that $W(\vec{o}) \leq 2 \cdot W(\vec{s})$. Similarly to the proof for cost-sharing games that we saw in class, consider each player i switching from s_i to o_i while the other players continue to play according to $\vec{s}_{-i}$. Because $\vec{s}$ is a Nash equilibrium, we have $u_i(\vec{s}) \geq u_i(o_i, \vec{s}_{-i})$ for each player i . Summing over all players,

$$W(\vec{s}) = \sum_{i=1}^n u_i(\vec{s}) \geq \sum_{i=1}^n u_i(o_i, \vec{s}_{-i}).$$

Next, we lower bound the right-hand side. Consider any player i such that no player chooses market o_i under $\vec{s}$, i.e., $o_i \notin M(\vec{s})$. If player i switches to market o_i , she becomes the only player in that market, so $u_i(o_i, \vec{s}_{-i}) = v_{o_i}$. Every market $j \in M(\vec{o}) \setminus M(\vec{s})$ is chosen by at least one such player under $\vec{o}$. Since rewards are nonnegative, we get

$$\sum_{i=1}^n u_i(o_i, \vec{s}_{-i}) \geq \sum_{j \in M(\vec{o}) \setminus M(\vec{s})} v_j = W(\vec{o}) - \sum_{j \in M(\vec{o}) \cap M(\vec{s})} v_j \geq W(\vec{o}) - W(\vec{s}),$$

where the last inequality holds because $\sum_{j \in M(\vec{o}) \cap M(\vec{s})} v_j \leq \sum_{j \in M(\vec{s})} v_j = W(\vec{s})$. Combining the two displayed inequalities, we get $W(\vec{s}) \geq W(\vec{o}) - W(\vec{s})$, i.e., $W(\vec{o}) \leq 2 \cdot W(\vec{s})$. Because this holds for every pure Nash equilibrium $\vec{s}$, the price of anarchy is at most 2.

PS: Unlike in the proof for cost-sharing games, we cannot compare the reward of each player under $\vec{s}$ and $\vec{o}$ individually. For example, in the game constructed in part (c) below, player 1 receives reward 1 under the socially optimal strategy profile but only $1/n$ under the pure Nash equilibrium. This is why we sum the inequalities over all players before comparing.

(c) Consider the game with n players and $m = n$ markets, where $v_1 = 1$ and $v_j = 1/n$ for $j \in \{2, \dots, n\}$.

- If all players choose market 1, each player receives reward $1/n$. A player switching to any market $j \geq 2$ would become the only player in that market and receive reward $1/n$ as well, which is not an improvement. Hence, this is a pure Nash equilibrium, and its social welfare is 1.
- If each player j chooses market j , the social welfare is $1 + (n-1)/n = 2 - 1/n$.

Hence, the optimal social welfare is at least $2 - 1/n$, while the social welfare in the worst pure Nash equilibrium is at most 1. Thus, the price of anarchy is at least $2 - 1/n$, which approaches 2 as n grows.