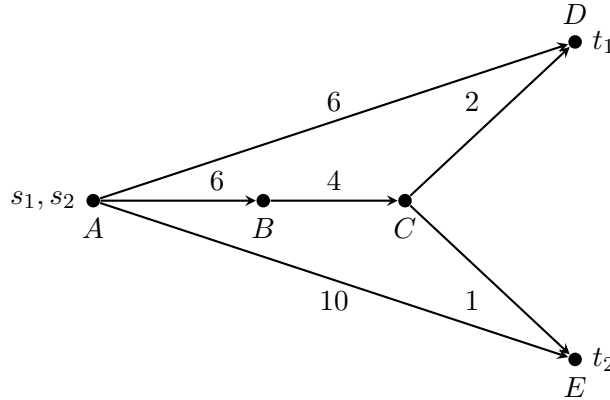


CSC304 Fall'26

Tutorial 3

Q1 Cost-sharing games

Consider the cost-sharing game shown in the figure below. There are two players, one wishing to go from s_1 to t_1 and the other from s_2 to t_2 (both s_1 and s_2 are the same node A). The costs of the edges are indicated in the figure.



- (a) Find a socially optimal strategy profile. What is the optimal social cost? Is this strategy profile a Nash equilibrium?
- (b) Find all pure Nash equilibria of this game and social costs under these equilibria.
- (c) What are the prices of anarchy and stability (with respect to pure Nash equilibria) of this game?

Q2 Price of stability in cost-sharing games

In class, we saw that the price of stability of any cost-sharing game is $O(\log n)$, where n is the number of players. Construct a cost-sharing game with n players in which the price of stability is $\Omega(\log n)$, thus establishing tightness of this bound.

Q3 Potential Functions

Recall that a potential function for a game assigns a real number $\Phi(\vec{s})$ to every pure strategy profile $\vec{s}$ such that whenever a single player changes her strategy, the change in the potential function is equal to the change in the cost to that player. That is, for every pure strategy profile $\vec{s}$, player i , and strategy s'_i of player i ,

$$c_i(s'_i, \vec{s}_{-i}) - c_i(\vec{s}) = \Phi(s'_i, \vec{s}_{-i}) - \Phi(\vec{s}).$$

In both games below, the first number in each cell is the cost to P1 and the second number is the cost to P2. Each player wants to minimize her cost.

(a) Construct a potential function for the following game by writing down the value of the potential function in each cell. Which cell minimizes your potential function? Is it a pure Nash equilibrium? Are there any other pure Nash equilibria of the game?

$P_1 \backslash P_2$	L	R
T	(3,5)	(7,8)
M	(5,6)	(3,3)
B	(7,6)	(5,3)

(b) Show that the following game does not have a potential function, even though it has a pure Nash equilibrium.

$P_1 \backslash P_2$	L	R
T	(2,2)	(3,1)
B	(1,4)	(2,2)

Q4 Market Sharing Games

Consider the following game. There are n players and m markets. Each market $j \in \{1, \dots, m\}$ has a value $v_j > 0$. Each player chooses a single market to enter, and the value of each market is split equally among all players who enter it. Formally, given a pure strategy profile $\vec{s} = (s_1, \dots, s_n)$, where $s_i \in \{1, \dots, m\}$ is the market chosen by player i , let $n_j(\vec{s})$ denote the number of players choosing market j . Then, the reward to player i is $u_i(\vec{s}) = v_{s_i}/n_{s_i}(\vec{s})$. The social welfare of $\vec{s}$ is the total reward of the players, $W(\vec{s}) = \sum_{i=1}^n u_i(\vec{s})$. Throughout this question, the price of anarchy is with respect to pure Nash equilibria.

(a) A potential function for a game with rewards is defined analogously to that for a game with costs: for every pure strategy profile $\vec{s}$, player i , and strategy s'_i of player i , we need $u_i(s'_i, \vec{s}_{-i}) - u_i(\vec{s}) = \Phi(s'_i, \vec{s}_{-i}) - \Phi(\vec{s})$. Construct a potential function for this game, and conclude that the game always has a pure Nash equilibrium.

(b) Show that the price of anarchy of this game is at most 2.

[Hint: Fix a pure Nash equilibrium $\vec{s}$ and a socially optimal strategy profile $\vec{o}$. For each player i , use the fact that she cannot increase her reward by switching from s_i to o_i , and sum over all players. Then, consider the markets chosen by at least one player under $\vec{o}$ but by no player under $\vec{s}$.]

(c) Show that the bound in part (b) is almost tight: for every n , construct a game with n players whose price of anarchy is at least $2 - 1/n$.