

CSC304 Fall'26

Solutions to Tutorial 2

Q1 Iterated Elimination of Dominated Actions

Recall elimination of strictly/weakly dominated actions: we consider the given game, find a strictly/weakly dominated action s_i of player i , and remove it from the game. In case of iterated elimination of strictly/weakly dominated actions, we repeat this process until there are no strictly/weakly dominated actions left in the game.

- (a) Show that elimination of a strictly dominated action cannot add or remove Nash equilibria, i.e., the set of Nash equilibria is preserved. (Note: This is why in iterated elimination of strictly dominated actions, the order in which we remove strictly dominated actions does not matter.)
- (b) Show that elimination of a weakly dominated action cannot add Nash equilibria (prove this), but can remove existing Nash equilibria (give an example).
- (c) Produce an example game in which the order of elimination of weakly dominated actions can affect which Nash equilibria remain in the resulting game.

Solution to Q1

We say that a mixed strategy s_i *uses* an action t_i if t_i is played with a positive probability under s_i .

- (a) This follows from part (b), where we show that elimination of a weakly dominated action cannot add Nash equilibria, because a strictly dominated action is also a weakly dominated action.

To see that it cannot remove Nash equilibria, suppose we eliminate an action t_i of player i that is strictly dominated by action t_i^* . Consider a Nash equilibrium $\vec{s}$ of the original game. We show that it is also a Nash equilibrium in the resulting game.

First, we observe that $\vec{s}$ is still well-defined after eliminating t_i . That is, s_i does not use t_i . This is true because if s_i used t_i , replacing t_i with t_i^* in the mixed strategy would yield a strictly higher reward, contradicting the fact that $\vec{s}$ is a NE.

Next, we note that for each player j , s_j is still the best response to $\vec{s}_{-j}$. This is also easy to see because a) s_j was the best response in the original game, b) the strategies used by the other players do not change, and c) the strategy space in the new game is a subset of the strategy space in the original game.

Hence, elimination of a strictly dominated action cannot remove Nash equilibria either.

- (b) Suppose for player i , action t_i is weakly dominated by action t_i^* and we eliminate t_i . We argue that no new Nash equilibria are created.

For contradiction, suppose $\vec{s}$ is a newly created (possibly mixed) Nash equilibrium. Hence, s_i cannot use action t_i , which is already eliminated. Because $\vec{s}$ was not an equilibrium before the elimination, there must exist an agent j and a strategy s'_j such that $u_j(s'_j, \vec{s}_{-j}) > u_j(\vec{s})$.

If $j \neq i$, or if $j = i$ and s'_i does not use t_i , then s'_j is still well-defined in the resulting game, and agent j can increase her reward by replacing s_j with s'_j , contradicting the fact that $\vec{s}$ is a NE.

If $j = i$ and s'_i uses t_i , consider a strategy s_i^* in which we replace playing t_i with playing t_i^* . Because t_i^* weakly dominates t_i , we have

$$u_i(s_i^*, \vec{s}_{-i}) \geq u_i(s'_i, \vec{s}_{-i}) > u_i(\vec{s}).$$

Because s_i^* does not use t_i , it is well-defined in the resulting game, and allows player i to increase her reward, again contradicting the fact that $\vec{s}$ is a NE.

Hence, elimination of a weakly dominated action cannot add a Nash equilibrium.

To see that it can remove Nash equilibria, consider the following game.

$P_1 \backslash P_2$	L	R
T	(1,1)	(0,0)
M	(1,1)	(2,1)
B	(0,0)	(2,1)

It is easy to see that (T,L) is a pure Nash equilibrium of the game. However, T is a weakly dominated by M, and removing T would remove this Nash equilibrium.

(c) Consider the game in the previous part.

We can remove T (weakly dominated by M) and then L (weakly dominated by R after T is removed). In the resulting game, (T,L) is not a Nash equilibrium (both actions no longer exist).

Or we can also remove B (weakly dominated by M) and then R (weakly dominated by L after B is removed). In the resulting game, (T,L) is a pure Nash equilibrium.

Hence, the order of removal of weakly dominated actions affects the set of Nash equilibria in the resulting game.

Q2 Computing Nash Equilibria

Consider the following normal-form game between two players. Each player has two actions. Payoffs for the row player P1 are indicated first in each cell, and payoffs for the column player P2 are second.

$P1 \backslash P2$	t_1	t_2
s_1	(3,6)	(1,1)
s_2	(1,2)	(2,4)

Find all mixed Nash equilibria of this game (including the pure ones).

Solution to Q2

Let us divide the possible equilibria into three cases.

1. *Neither player randomizes.* In this case, we want to identify pure Nash equilibria. Hence, we draw the best-response diagram, which is as follows.

$P1 \backslash P2$	t_1	t_2
s_1	(3,6) ← (1,1)	
s_2	(1,2) → (2,4)	

We can see that the only cells with no outgoing arrows are (s_1, t_1) and (s_2, t_2) . Hence, these are the pure Nash equilibria.

2. *One player doesn't randomize, the other does.* It is easy to see that if one player doesn't randomize, the two actions of the other player do not yield equal rewards (e.g., if P1 plays s_1 , then the rewards of P2 for playing t_1 and t_2 are different.)

The indifference principle says that the other player can randomize between two actions only if her rewards from the two actions are equal. Hence, the other player cannot randomize in a Nash equilibrium.

3. *Both players randomize.* Suppose P1 plays s_1 with probability p and s_2 with probability $1 - p$, and that P2 plays t_1 with probability q and t_2 with probability $1 - q$. Because both players randomize, by the indifference principle, each player must have equal rewards for both her actions given the mixed strategy of the other player.

- Reward of P1: Fixing the mixed strategy of P2, the rewards of P1 for playing s_1 and s_2 must be equal. Hence,

$$\text{Reward from } s_1 = q \cdot 3 + (1 - q) \cdot 1$$

$$\text{Reward from } s_2 = q \cdot 1 + (1 - q) \cdot 2$$

$$q \cdot 3 + (1 - q) \cdot 1 = q \cdot 1 + (1 - q) \cdot 2 \Rightarrow q = \frac{1}{3}.$$

- Reward of P2: Fixing the mixed strategy of P1, the rewards of P2 for playing t_1 and t_2 must be equal. Hence,

$$\text{Reward from } t_1 = p \cdot 6 + (1 - p) \cdot 2$$

$$\text{Reward from } t_2 = p \cdot 1 + (1 - p) \cdot 4$$

$$p \cdot 6 + (1 - p) \cdot 2 = p \cdot 1 + (1 - p) \cdot 4 \Rightarrow p = \frac{2}{7}.$$

Thus, the game has precisely three Nash equilibria: two pure equilibria (s_1, t_1) and (s_2, t_2) , and a mixed Nash equilibrium in which P1 plays s_1 w.p. $2/7$ and P2 plays t_1 w.p. $1/3$.

PS: Note that, interestingly, the probabilities with which P1 plays her two actions in the mixed Nash equilibrium depend only on the rewards of P2 (and not on the rewards of P1), and vice-versa.

Q3 Cunning Airlines

Recall the cunning airlines game from class. There are two players, say P1 and P2. Each player can report an integer in $\{2, 3, \dots, 99\}$. If P1 reports s and P2 reports t , their rewards are determined as follows.

- If $s = t$, both players receive s .
- If $s < t$, P1 receives $s + 2$ and P2 receives $s - 2$.
- If $t < s$, P1 receives $t - 2$ and P2 receives $t + 2$.

Find a Nash equilibrium of this game, and prove that it is the unique Nash equilibrium (that is, there are no other mixed Nash equilibria).

Solution to Q3

Given the observations in Q1, it might be tempting to reduce the size of the game by iteratively eliminating dominated actions. Indeed, for instance, reporting 99 is dominated by reporting 98, but the domination is weak. Hence, we cannot remove this action directly while leveraging the observation in Q1. We use a slightly different argument below.

First, we note that each player reporting 2 (deterministically) is trivially a pure Nash equilibrium of the game.

We now show that it is unique among all (possibly mixed) Nash equilibria. Consider a (possibly mixed) Nash equilibrium (s_1, s_2) . Let M be the highest number used by s_1 or s_2 . If we show that $M = 2$, then it follows that the aforementioned pure NE is the unique equilibrium of the game.

Suppose for contradiction that $M > 2$. Without loss of generality, suppose s_1 (P1's strategy) uses M . Let T be the largest action used by s_2 (P2's strategy). Note that $T \leq M$.

If $T = 2$, then P2 is reporting 2 deterministically. In this case, P1 has only two possible rewards: a reward of 2 when reporting 2, and a reward of 0 when reporting any other number. Hence, by the

indifference principle, P1 must report 2 deterministically, which contradicts our assumption $M > 2$.

If $T > 2$, consider a strategy s'_1 in which P1 replaces playing M with playing $T - 1$. We argue that this strictly increases the reward of P1, contradicting the fact that (s_1, s_2) is a NE. Consider the reward to P1 from playing M and from playing $T - 1$ for different possible reports of P2.

	Reward to P1 under M	Reward to P1 under $T - 1$
P2 reports $s < T - 1$	$s - 2$	$s - 2$
P2 reports $T - 1$	$T - 3$	$T - 1$
P2 reports T	$\leq T$	$T + 1$

Hence, playing $T - 1$ yields at least as much reward to P1 as playing M does for every report of P2. In at least one case that occurs with a positive probability (P2 playing T), the reward is strictly higher. Hence, given the mixed strategy s_2 of P2, $T - 1$ yields a strictly higher expected reward to P1 than M . Hence, s'_1 yields a strictly higher reward to P1 than s_1 , as desired.

Q4 Guess $3/4^{\text{th}}$ of the average

Recall the game we played in class called “guess $2/3^{\text{rd}}$ of the average”. Consider a variant of this game played among three players.

Each of Alice, Bob, and Charlie picks a number in the set $\{1, 2, 3, 4\}$. The player whose number is closest to $3/4^{\text{th}}$ of the average of the three numbers wins the game (i.e., gets reward 1) and the other players lose the game (i.e., get reward 0). If multiple players are tied for being the closest to $3/4^{\text{th}}$ of the average, they share the reward equally (i.e., if two players are tied, each of them gets reward $1/2$, and if all three players are tied, each of them gets reward $1/3$), and the remaining player (if any) gets reward 0.

Find all Nash equilibria of this game.

Solution to Q4

Note: While this game cannot be solved via iterated elimination of strictly dominated actions (because no action strictly dominates another action), one can observe that playing 4 is strictly dominated by the mixed strategy of playing 1 with probability 0.6 and playing 2 with probability 0.4. For the same reason that we can eliminate actions strictly dominated by other actions without affecting Nash equilibria, we can also eliminate actions strictly dominated by mixed strategies. Applying this would solve the presented game much more easily. But I’m presenting a “derivation from first principles” here for those who may prefer it.

Solution: We prove that each player choosing 1 is the only mixed Nash equilibrium, even in the general game where, for some positive integer t , each player’s action set is $\{1, 2, \dots, t\}$. We write $u_i(a, \vec{s}_{-i})$ for the expected reward of player i when she chooses a and the other two players play according to $\vec{s}_{-i}$. We begin with two observations about pure strategy profiles.

Fact 1: In any pure strategy profile in which the three numbers are not all equal, every player choosing the highest of the three numbers receives reward 0. To see this, let m and M denote the lowest and the highest of the three numbers, and let ℓ denote the remaining number, so that

$m \leq \ell \leq M$ and $m < M$. Then, $3/4^{\text{th}}$ of the average of the three numbers is $T = (m + \ell + M)/4$. Because $\ell \leq M < M + m$, we have $T < (m + M)/2$, which implies $M - T > T - m$. Also, $M - T > m - T$ because $M > m$. Hence, $|M - T| = M - T > |m - T|$, i.e., the player choosing m is strictly closer to T than every player choosing M , so every player choosing M receives reward 0.

Fact 2: In any pure strategy profile in which two players choose the same number a and the third player chooses a number $b \geq a$, both players choosing a receive a positive reward. If $b = a$, then all three players are tied, and each receives reward $1/3$. If $b > a$, then by Fact 1, the player choosing b receives reward 0, so the two players choosing a , who are equally close to $3/4^{\text{th}}$ of the average, are the closest, and each receives reward $1/2$.

Consider any mixed Nash equilibrium $\vec{s}$. Let k^* denote the highest number assigned a positive probability under any of s_1 , s_2 , and s_3 . Without loss of generality, suppose s_1 assigns a positive probability to k^* . We want to prove that $k^* = 1$.

Suppose for contradiction that $k^* > 1$. Due to the indifference principle, k^* must be a best response for player 1 given $\vec{s}_{-1}$:

$$u_1(k^*, \vec{s}_{-1}) \geq u_1(i, \vec{s}_{-1}), \quad \forall i \in \{1, \dots, t\}. \quad (1)$$

Players 2 and 3 choose numbers in $\{1, \dots, k^*\}$ with probability 1. Hence, whenever player 1 chooses k^* , it is the highest of the three numbers, so by Fact 1, player 1 receives reward $1/3$ if players 2 and 3 both choose k^* , and reward 0 otherwise. Let p denote the probability that players 2 and 3 both choose k^* under $\vec{s}_{-1}$. Then, $u_1(k^*, \vec{s}_{-1}) = p/3$. We consider two cases:

Case 1: Suppose $p > 0$. In this case, we prove that $u_1(k^* - 1, \vec{s}_{-1}) > u_1(k^*, \vec{s}_{-1})$, which contradicts Equation (1). When player 1 chooses $k^* - 1$ and players 2 and 3 both choose k^* , by Fact 1, players 2 and 3 receive reward 0, so player 1 receives reward 1. Since rewards are nonnegative, we have $u_1(k^* - 1, \vec{s}_{-1}) \geq p > p/3 = u_1(k^*, \vec{s}_{-1})$.

Case 2: Suppose $p = 0$, so $u_1(k^*, \vec{s}_{-1}) = 0$. In this case, we find an action a with $u_1(a, \vec{s}_{-1}) > 0$, which also contradicts Equation (1). Fix numbers j and j' such that the probability q that player 2 chooses j and player 3 chooses j' is positive, and let $a = \min(j, j')$. When players 1, 2, and 3 choose a , j , and j' , respectively, two players choose a and the third player chooses $\max(j, j') \geq a$, so by Fact 2, player 1 receives a positive reward r . Since rewards are nonnegative, we have $u_1(a, \vec{s}_{-1}) \geq q \cdot r > 0$.

Hence, our original assumption that $k^* > 1$ is wrong. Thus, $k^* = 1$, i.e., each player chooses 1 with probability 1. Finally, this is indeed a Nash equilibrium: each player receives reward $1/3$, and by Fact 1, a player who deviates to any number greater than 1 receives reward 0. Hence, the only mixed Nash equilibrium is for each player to choose 1 with probability 1.

Q5 Russell Crowe was wrong!

Watch the scene from the movie *A Beautiful Mind* that purports to explain what a Nash equilibrium is (it's easy to find this on YouTube). This can be modeled as a game with four players (the men) and each player having five actions (corresponding to the women). Explain why the solution proposed by Russell Crowe (playing John Nash) is *not* a Nash equilibrium.

Solution to Q5

The game portrayed is the following. There are four players in the game, which are the four men. There are five women: a special woman (let's call her X) and her four friends. Each man has five actions corresponding to asking the five women to dance. If multiple men ask the same woman, they all receive reward 0, but if a single man asks a woman, he receives a positive reward, which is higher for woman X and lower for her friends.

Russell Crowe claims that it is a Nash equilibrium for the four men to ask the four friends, leaving X alone. This is not a Nash equilibrium because in this strategy profile, one of the men will have an incentive to switch to asking X .