

Question 1. Nash Equilibria [35 MARKS]

Consider the following normal-form game involving two players. The row player (Alice) and the column player (Bob) have four actions each. In each cell of the payoff matrix, the first number indicates the payoff to the row player (Alice) and the second number indicates the payoff to the column player (Bob). In this game, the payoffs are negative, so the players actually incur costs.

	A	B	C	D
W	(-5, -8)	(-5, -4)	(-8, -7)	(-1, -7)
X	(-4, -7)	(-3, -7)	(-7, -8)	(-6, -8)
Y	(-7, -5)	(-6, -4)	(-5, -5)	(-3, -4)
Z	(-3, -5)	(-7, -5)	(-6, -6)	(-5, -6)

Part (a) [5 MARKS]

If Alice plays *W*, what is Bob's best response? You do not need to justify your answer.

Solution: *B*

Part (b) [5 MARKS]

If Bob plays *C*, what is Alice's best response? You do not need to justify your answer.

Solution: *Y*

Part (c) [5 MARKS]

It is possible to iteratively eliminate strictly dominated actions and reduce the game to a 2×2 game in which only two actions of each player remain. State which actions you would remove and in which order. You do not need to justify your answer.

Solution: *C, Y, D, W*

Part (d) [5 MARKS]

In the reduced 2×2 game produced in part (c), find all (if any) pure Nash equilibria. You only need to identify the equilibria; no explanation is needed.

Solution: *(Z, A)* and *(X, B)*

Part (e) [10 MARKS]

In the reduced 2×2 game produced in part (c), find all mixed Nash equilibria, apart from those already found in part (d), using the indifference principle. Show your calculation for full marks.

Solution: There are three remaining cases.

- *Bob does not randomize, Alice does.* It is easy to check that Alice's rewards for her two actions are different for each action of Bob. Hence, there are no NE in this case by the indifference principle.
- *Alice does not randomize, Bob does.* Indeed, for each action of Alice, Bob is indifferent between his two actions.
 - Suppose Alice plays *X* and Bob plays *A* w.p. q and *B* w.p. $1 - q$, where $q \in (0, 1)$. Then, for Alice to prefer *X*, we need $-4q - 3(1 - q) \geq -3q - 7(1 - q) \Leftrightarrow 4(1 - q) \geq q \Leftrightarrow q \leq 4/5$. So $q \in (0, 4/5]$.
 - Suppose Alice plays *Z* and Bob plays *A* w.p. q and *B* w.p. $1 - q$, where $q \in (0, 1)$. Symmetrically, we need $q \geq 4/5$ in this case, so $q \in [4/5, 1)$.
- *Alice and Bob both randomize.* Suppose Alice plays *X* w.p. p and *Z* w.p. $1 - p$, while Bob plays *A* w.p. q and *B* w.p. $1 - q$, where $p, q \in (0, 1)$. By the indifference principle for Alice, we need $-4q - 3(1 - q) = -3q - 7(1 - q) \Leftrightarrow q = 4/5$. Bob is always indifferent between his two actions, so any value of p works.

Hence, the remaining equilibria are as follows. The first two numbers are probabilities of Alice playing *X* and *Z*, respectively, and the last two numbers are probabilities of Bob playing *A* and *B*, respectively.

$$\begin{aligned}
 & \{((1, 0), (q, 1 - q)) : q \in (0, 4/5]\} \\
 & \cup \{((0, 1), (q, 1 - q)) : q \in [4/5, 1)\} \\
 & \cup \{((p, 1 - p), (4/5, 1/5)) : p \in (0, 1)\}.
 \end{aligned}$$

Part (f) [5 MARKS]

If the original 4x4 game is treated as a Stackelberg game in which first Alice commits to a pure strategy and then Bob responds, which pure strategy should Alice commit to, what would Bob respond with, and what would be Alice's reward in that case? No explanation is needed.

Solution: As the question is stated, any of the following answers can be accepted for (Alice's strategy, Bob's response, Alice's reward).

- (X,A,-4), (X,B,-3)
- (Y,B,-6), (Y,D,-3)
- (Z,A,-3), (Z,B,-7)

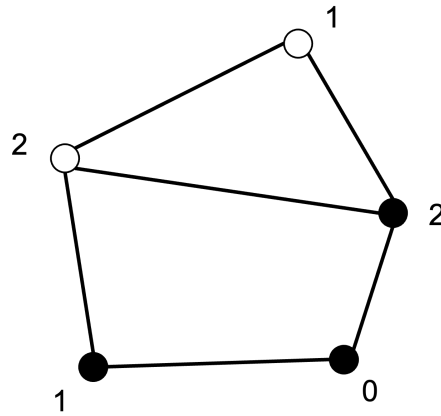
If we apply the assumption from class that the follower breaks ties in favor of the leader, then it narrows down to the following answers.

- (X,B,-3)
- (Y,D,-3)
- (Z,A,-3)

Question 2. Price of Anarchy [15 MARKS]

Consider a *colouring game* is played on an undirected graph $G = (V, E)$ with $|E| \geq 1$, where every vertex $v \in V$ is a player who can choose a pure strategy $s_v \in \{\text{black}, \text{white}\}$. The reward to each player v in this pure strategy profile $\vec{s}$ is the number of her neighbours with colour opposite to hers, i.e., $u_v(\vec{s}) = |\{v' \in V : (v, v') \in E \wedge s_{v'} \neq s_v\}|$. The social welfare is defined as the sum of rewards to all players, i.e., $W(\vec{s}) = \sum_{v \in V} u_v(\vec{s})$.

For example, in the graph drawn below, each player's vertex is shaded by her chosen colour and her reward in this pure strategy profile is indicated next to her vertex.

**Part (a)** [10 MARKS]

Prove that $W(\vec{s}) \geq |E|$ under every pure Nash equilibrium of the game.

Solution: Let d_v denote the degree of vertex v .

Under any pure Nash equilibrium $\vec{s}$, at least half of v 's neighbors must have colour opposite to hers, otherwise v would have an incentive to switch colors. Hence, $u_v(\vec{s}) \geq d_v/2$ for each $v \in V$.

Summing over all $v \in V$ and using the fact that $\sum_{v \in V} d_v = 2|E|$, we get $W(\vec{s}) = \sum_{v \in V} u_v(\vec{s}) \geq (1/2) \sum_{v \in V} d_v = |E|$.

Part (b) [5 MARKS]

Use the above argument to prove that the price of anarchy of the game with respect to pure Nash equilibria is at most 2.

Solution: Let $\vec{s}$ be any pure Nash equilibrium and $\vec{s}^*$ be a social optimum. Assume $|E| \geq 1$ (otherwise all welfares are 0 and the ratio is undefined).

By part (a), $W(\vec{s}) \geq |E| \geq 1$. On the other hand, $W(\vec{s}^*) = \sum_{v \in V} u_v(s^*) \leq \sum_{v \in V} d_v = 2|E|$. Hence, $\frac{W(\vec{s}^*)}{W(\vec{s})} \leq 2$.

Since this holds for any pure Nash equilibrium $\vec{s}$, the price of anarchy with respect to such equilibria is at most 2.

Bonus Question. Approximate Nash Equilibria [10 MARKS]

In a 2-player game, let $\vec{s} = (s_1, s_2)$ denote a strategy profile, where s_1 and s_2 are the (possibly mixed) strategies of players 1 and 2, respectively. Let $\pi_1(\vec{s}), \pi_2(\vec{s}) \in [0, 1]$ be the expected payoffs to players 1 and 2 respectively under $\vec{s}$.

For $\epsilon \geq 0$, we say that $\vec{s} = (s_1, s_2)$ is an ϵ -Nash equilibrium if no player can increase her reward by more than ϵ by switching to another strategy. That is,

$$\begin{aligned} \pi_1(s_1, s_2) &\geq \pi_1(s'_1, s_2) - \epsilon, \forall s'_1, \text{ and} \\ \pi_2(s_1, s_2) &\geq \pi_2(s_1, s'_2) - \epsilon, \forall s'_2. \end{aligned}$$

Note that when $\epsilon = 0$, this is simply the Nash equilibrium definition.

Now, construct a specific strategy profile $\vec{s} = (s_1, s_2)$ as follows.

- Choose any action a_1 of player 1.
- Let a_2 be a best response of player 2 to player 1 playing a_1 (i.e., $\pi_2(a_1, a_2) \geq \pi_2(a_1, t)$ for all actions t of player 2).
- Let a'_1 be a best response of player 1 to player 2 playing a_2 (i.e. $\pi_1(a'_1, a_2) \geq \pi_1(t, a_2)$ for all actions t of player 1).
- Let s_1 be the mixed strategy where player 1 plays a_1 and a'_1 with probability 1/2 each.
- Let s_2 be the pure strategy where player 2 plays a_2 deterministically.

Show that $\vec{s} = (s_1, s_2)$ must be a (1/2)-Nash equilibrium of the game.

Solution: Let s'_1 be an arbitrary mixed strategy of player 1.

$$\begin{aligned} \pi_1(s'_1, s_2) - \pi_1(s_1, s_2) &= \pi_1(s'_1, a_2) - \left(\frac{1}{2} \pi_1(a_1, a_2) + \frac{1}{2} \pi_1(a'_1, a_2) \right) \\ &= \frac{1}{2} \cdot (\pi_1(s'_1, a_2) - \pi_1(a_1, a_2)) + \frac{1}{2} \cdot (\pi_1(s'_1, a_2) - \pi_1(a'_1, a_2)) \\ &\leq \frac{1}{2} \cdot (\pi_1(s'_1, a_2) - \pi_1(a_1, a_2)) \leq \frac{1}{2}, \end{aligned}$$

where the third transition holds because a'_1 is P1's best response to P2 playing a_2 and gives her at least as much reward as playing s'_1 , and the last transition holds simply because all rewards are in $[0, 1]$.

Similarly, let s'_2 be an arbitrary mixed strategy of player 2.

$$\begin{aligned} \pi_2(s_1, s'_2) - \pi_2(s_1, s_2) &= \left(\frac{1}{2} \pi_2(a_1, s'_2) + \frac{1}{2} \pi_2(a'_1, s'_2) \right) - \left(\frac{1}{2} \pi_2(a_1, a_2) + \frac{1}{2} \pi_2(a'_1, a_2) \right) \\ &= \frac{1}{2} \cdot (\pi_2(a_1, s'_2) - \pi_2(a_1, a_2)) + \frac{1}{2} \cdot (\pi_2(a'_1, s'_2) - \pi_2(a'_1, a_2)) \\ &\leq \frac{1}{2} \cdot (\pi_2(a'_1, s'_2) - \pi_2(a'_1, a_2)) \leq \frac{1}{2}, \end{aligned}$$

where the third transition holds because a_2 is P2's best response to P1 playing a_1 and gives her at least as much reward as playing s'_2 , and the last transition holds simply because all rewards are in $[0, 1]$.