

UNIVERSITY OF TORONTO  
Faculty of Arts & Science

MIDTERM 1 EXAMINATION

CSC 304 H1F

Duration: 50 minutes

Aids Allowed: One 8.5"×11" sheet of  
handwritten notes on one side.

*Do **not** turn this page until  
you have received the signal to start.*  
In the meantime, write your name and student  
number below (please do this now!) and  
*carefully* read *all* the information  
on the rest of this page.

First (Given) Name(s):

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Last (Family) Name(s):

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- This midterm examination consists of 2 questions and 1 bonus question on 8 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the examination is complete.*
- Answer each question directly on the examination paper, in the space provided, and use a “blank” page for rough work. If you need more space for one of your solutions, use one of the “blank” pages and *indicate clearly the part of your work that should be marked.*
- As a student, you help create a fair and inclusive writing environment. If you possess an unauthorized aid during an exam, you may be charged with an academic offence.

MARKING GUIDE

N° 1: \_\_\_\_\_/35

N° 2: \_\_\_\_\_/15

BONUS: \_\_\_\_\_/10

TOTAL: \_\_\_\_\_/50

STUDENTS MUST HAND IN ALL EXAMINATION MATERIALS AT THE END

**Question 1.** Nash Equilibria [35 MARKS]

Consider the following normal-form game involving two players. The row player (Alice) and the column player (Bob) have four actions each. In each cell of the payoff matrix, the first number indicates the payoff to the row player (Alice) and the second number indicates the payoff to the column player (Bob). In this game, the payoffs are negative, so the players actually incur costs.

|          | <i>A</i> | <i>B</i> | <i>C</i> | <i>D</i> |
|----------|----------|----------|----------|----------|
| <i>W</i> | (-5, -8) | (-5, -4) | (-8, -7) | (-1, -7) |
| <i>X</i> | (-4, -7) | (-3, -7) | (-7, -8) | (-6, -8) |
| <i>Y</i> | (-7, -5) | (-6, -4) | (-5, -5) | (-3, -4) |
| <i>Z</i> | (-3, -5) | (-7, -5) | (-6, -6) | (-5, -6) |

**Part (a)** [5 MARKS]

If Alice plays *W*, what is Bob's best response? You do not need to justify your answer.

**Part (b)** [5 MARKS]

If Bob plays *C*, what is Alice's best response? You do not need to justify your answer.

**Part (c)** [5 MARKS]

It is possible to iteratively eliminate strictly dominated actions and reduce the game to a  $2 \times 2$  game in which only two actions of each player remain. State which actions you would remove and in which order. You do not need to justify your answer.

**Part (d)** [5 MARKS]

In the reduced  $2 \times 2$  game produced in part (c), find all (if any) pure Nash equilibria. You only need to identify the equilibria; no explanation is needed.

**Part (e)** [10 MARKS]

In the reduced  $2 \times 2$  game produced in part (c), find all mixed Nash equilibria, apart from those already found in part (d), using the indifference principle. Show your calculation for full marks.

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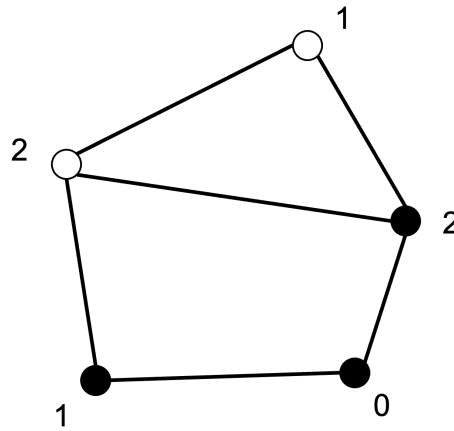
**Part (f)** [5 MARKS]

If the original 4x4 game is treated as a Stackelberg game in which first Alice commits to a pure strategy and then Bob responds, which pure strategy should Alice commit to, what would Bob respond with, and what would be Alice's reward in that case? No explanation is needed.

**Question 2.** Price of Anarchy [15 MARKS]

Consider a *colouring game* is played on an undirected graph  $G = (V, E)$  with  $|E| \geq 1$ , where every vertex  $v \in V$  is a player who can choose a pure strategy  $s_v \in \{\text{black}, \text{white}\}$ . The reward to each player  $v$  in this pure strategy profile  $\vec{s}$  is the number of her neighbours with colour opposite to hers, i.e.,  $u_v(\vec{s}) = |\{v' \in V : (v, v') \in E \wedge s_{v'} \neq s_v\}|$ . The social welfare is defined as the sum of rewards to all players, i.e.,  $W(\vec{s}) = \sum_{v \in V} u_v(\vec{s})$ .

For example, in the graph drawn below, each player's vertex is shaded by her chosen colour and her reward in this pure strategy profile is indicated next to her vertex.

**Part (a)** [10 MARKS]

Prove that  $W(\vec{s}) \geq |E|$  under every pure Nash equilibrium of the game.

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**Part (b)** [5 MARKS]

Use the above argument to prove that the price of anarchy of the game with respect to pure Nash equilibria is at most 2.

**Bonus Question.** Approximate Nash Equilibria [10 MARKS]

In a 2-player game, let  $\vec{s} = (s_1, s_2)$  denote a strategy profile, where  $s_1$  and  $s_2$  are the (possibly mixed) strategies of players 1 and 2, respectively. Let  $\pi_1(\vec{s}), \pi_2(\vec{s}) \in [0, 1]$  be the expected payoffs to players 1 and 2 respectively under  $\vec{s}$ .

For  $\epsilon \geq 0$ , we say that  $\vec{s} = (s_1, s_2)$  is an  $\epsilon$ -Nash equilibrium if no player can increase her reward by more than  $\epsilon$  by switching to another strategy. That is,

$$\begin{aligned}\pi_1(s_1, s_2) &\geq \pi_1(s'_1, s_2) - \epsilon, \forall s'_1, \text{ and} \\ \pi_2(s_1, s_2) &\geq \pi_2(s_1, s'_2) - \epsilon, \forall s'_2.\end{aligned}$$

Note that when  $\epsilon = 0$ , this is simply the Nash equilibrium definition.

Now, construct a specific strategy profile  $\vec{s} = (s_1, s_2)$  as follows.

- Choose any action  $a_1$  of player 1.
- Let  $a_2$  be a best response of player 2 to player 1 playing  $a_1$  (i.e.,  $\pi_2(a_1, a_2) \geq \pi_2(a_1, t)$  for all actions  $t$  of player 2).
- Let  $a'_1$  be a best response of player 1 to player 2 playing  $a_2$  (i.e.  $\pi_1(a'_1, a_2) \geq \pi_1(t, a_2)$  for all actions  $t$  of player 1).
- Let  $s_1$  be the mixed strategy where player 1 plays  $a_1$  and  $a'_1$  with probability 1/2 each.
- Let  $s_2$  be the pure strategy where player 2 plays  $a_2$  deterministically.

Show that  $\vec{s} = (s_1, s_2)$  must be a (1/2)-Nash equilibrium of the game.

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*Use the space on this “blank” page for scratch work, or for any solution that did not fit elsewhere.  
Clearly label each such solution with the appropriate question and part number.*

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