

Q1. Nash Equilibria (25 pts)

Consider the following normal-form game involving two players. The row player (Alice) and the column player (Bob) have four actions each. In each cell of the payoff matrix, the first number indicates the payoff to the row player (Alice) and the second number indicates the payoff to the column player (Bob). In this game, the payoffs are negative, so the players actually incur costs.

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
<i>W</i>	(−5, −8)	(−5, −4)	(−8, −7)	(−1, −7)
<i>X</i>	(−4, −7)	(−3, −7)	(−7, −8)	(−6, −8)
<i>Y</i>	(−7, −5)	(−6, −4)	(−5, −5)	(−3, −4)
<i>Z</i>	(−3, −5)	(−7, −5)	(−6, −6)	(−5, −6)

(a) [2.5 points] If Alice plays *W*, what is Bob's best response? You do not need to justify your answer.

Solution: *B*

(b) [2.5 points] If Bob plays *C*, what is Alice's best response? You do not need to justify your answer.

Solution: *Y*

(c) [5 points] It is possible to iteratively eliminate strictly dominated actions and reduce the game to a 2×2 game in which only two actions of each player remain. State which actions you would remove and in which order. You do not need to justify your answer.

Solution: *C, Y, D, W*

(d) [5 points] In the reduced 2×2 game produced in part (c), find all (if any) pure Nash equilibria. You only need to identify the equilibria; no explanation is needed.

Solution: (*Z, A*) and (*X, B*)

(e) [10 points] In the reduced 2×2 game produced in part (c), find all mixed Nash equilibria, apart from those already found in part (d), using the indifference principle. Show your calculation for full marks. In all Nash equilibria, clearly express which action each player is playing with what probability.

Solution: There are three remaining cases.

- *Bob does not randomize, Alice does.* It is easy to check that Alice's rewards for her two actions are different for each action of Bob. Hence, there are no NE in this case by the indifference principle.
- *Alice does not randomize, Bob does.* Indeed, for each action of Alice, Bob is indifferent between his two actions.
 - Suppose Alice plays *X* and Bob plays *A* w.p. q and *B* w.p. $1 - q$, where $q \in (0, 1)$. Then, for Alice to prefer *X*, we need $-4q - 3(1 - q) \geq -3q - 7(1 - q) \Leftrightarrow 4(1 - q) \geq q \Leftrightarrow q \leq 4/5$. So $q \in (0, 4/5]$.
 - Suppose Alice plays *Z* and Bob plays *A* w.p. q and *B* w.p. $1 - q$, where $q \in (0, 1)$. Symmetrically, we need $q \geq 4/5$ in this case, so $q \in [4/5, 1)$.
- *Alice and Bob both randomize.* Suppose Alice plays *X* w.p. p and *Z* w.p. $1 - p$, while Bob plays *A* w.p. q and *B* w.p. $1 - q$, where $p, q \in (0, 1)$. By the indifference principle for Alice, we need $-4q - 3(1 - q) = -3q - 7(1 - q) \Leftrightarrow q = 4/5$. Bob is always indifferent between his two actions, so any value of p works.

Hence, the remaining equilibria are as follows. The first two numbers are probabilities of Alice playing X and Z , respectively, and the last two numbers are probabilities of Bob playing A and B , respectively.

$$\begin{aligned} & \{((1, 0), (q, 1 - q)) : q \in (0, 4/5]\} \\ & \cup \{((0, 1), (q, 1 - q)) : q \in [4/5, 1)\} \\ & \cup \{((p, 1 - p), (4/5, 1/5)) : p \in (0, 1)\}. \end{aligned}$$

Q2. Approximate Nash Equilibria (10 pts)

In a 2-player normal-form game, let $\vec{s} = (s_1, s_2)$ denote a strategy profile, where s_1 and s_2 are the (possibly mixed) strategies of players 1 and 2, respectively. Let $\pi_1(\vec{s})$ and $\pi_2(\vec{s})$ denote the expected payoffs to players 1 and 2 respectively under $\vec{s}$. Assume that $\pi_1(\vec{s})$ and $\pi_2(\vec{s})$ are in the interval $[0, 1]$ for all $\vec{s}$.

For $\epsilon \geq 0$, we say that $\vec{s} = (s_1, s_2)$ is an ϵ -Nash equilibrium if no player can increase her reward by more than ϵ by switching to another strategy. That is,

$$\begin{aligned} \pi_1(s_1, s_2) &\geq \pi_1(s'_1, s_2) - \epsilon, \forall s'_1, \text{ and} \\ \pi_2(s_1, s_2) &\geq \pi_2(s_1, s'_2) - \epsilon, \forall s'_2. \end{aligned}$$

Note that when $\epsilon = 0$, this is simply the Nash equilibrium definition.

Now, construct a specific strategy profile $\vec{s} = (s_1, s_2)$ as follows.

- Let a_1 be an arbitrary action of player 1.
- Let a_2 be an action of player 2 that is best response to player 1 playing a_1 (i.e. $a_2 \in \arg \max_t \pi_2(a_1, t)$).
- Let a'_1 be an action of player 1 that is best response to player 2 playing a_2 (i.e. $a'_1 \in \arg \max_t \pi_1(t, a_2)$).
- Let s_1 be the mixed strategy where player 1 plays a_1 and a'_1 with probability $1/2$ each.
- Let s_2 be the pure strategy where player 2 plays a_2 deterministically.

Show that $\vec{s} = (s_1, s_2)$ must be a $(1/2)$ -Nash equilibrium of the game.

[Hint: Write down the expected payoff to each player in this strategy profile. Observe that each player is receiving the optimal payoff with some probability and a possibly suboptimal payoff with the remaining probability, given the strategy of the other player. Use this to show that a player cannot increase her payoff by more than $1/2$ by switching to any other strategy.]

Solution: As seen in class, a pure strategy best response exists (due to linearity of expectation), so a'_1 and a_2 are optimal even among mixed strategies. Let s'_1 be an arbitrary mixed strategy of player 1.

$$\begin{aligned} \pi_1(s'_1, s_2) - \pi_1(s_1, s_2) &= \pi_1(s'_1, a_2) - \left(\frac{1}{2} \pi_1(a_1, a_2) + \frac{1}{2} \pi_1(a'_1, a_2) \right) \\ &= \frac{1}{2} \cdot (\pi_1(s'_1, a_2) - \pi_1(a_1, a_2)) + \frac{1}{2} \cdot (\pi_1(s'_1, a_2) - \pi_1(a'_1, a_2)) \\ &\leq \frac{1}{2} \cdot (\pi_1(s'_1, a_2) - \pi_1(a_1, a_2)) \leq \frac{1}{2}, \end{aligned}$$

where the third transition holds because a'_1 is P1's best response to P2 playing a_2 and gives her at least as much reward as playing s'_1 , and the last transition holds simply because all rewards are in $[0, 1]$.

Similarly, let s'_2 be an arbitrary mixed strategy of player 2.

$$\begin{aligned}\pi_2(s_1, s'_2) - \pi_2(s_1, s_2) &= \left(\frac{1}{2}\pi_2(a_1, s'_2) + \frac{1}{2}\pi_2(a'_1, s'_2) \right) - \left(\frac{1}{2}\pi_2(a_1, a_2) + \frac{1}{2}\pi_2(a'_1, a_2) \right) \\ &= \frac{1}{2} \cdot (\pi_2(a_1, s'_2) - \pi_2(a_1, a_2)) + \frac{1}{2} \cdot (\pi_2(a'_1, s'_2) - \pi_2(a'_1, a_2)) \\ &\leq \frac{1}{2} \cdot (\pi_2(a'_1, s'_2) - \pi_2(a'_1, a_2)) \leq \frac{1}{2},\end{aligned}$$

where the third transition holds because a_2 is P2's best response to P1 playing a_1 and gives her at least as much reward as playing s'_2 , and the last transition holds simply because all rewards are in $[0, 1]$.

Q3. Stackelberg Equilibria (20 pts)

Consider the following game between two players. Again, the row player is Alice, the column player is Bob, the first number in each cell denotes the payoff to Alice, and the second number in each cell denotes the payoff to Bob.

	A	B
X	(3, 1)	(5, 0)
Y	(1, 0)	(2, 1)

(a) [5 points] Suppose both players have to choose their strategies simultaneously. Identify the unique Nash equilibrium of the game by iteratively eliminating strictly dominated actions. You only need to write down the actions removed, the order in which they are removed, and the unique Nash equilibrium; no further explanation is needed.

Solution: Actions removed in order: Y, B . Nash equilibrium: (X, A) .

(b) [5 points] Suppose Alice can commit first and decides to commit to a *pure* strategy. Which pure strategy would be best for her to commit to? What would be her reward in the resulting outcome? Justify your answer.

Solution: If Alice commits to X , then Bob will respond with A , giving Alice a reward of 3. If Alice commits to Y , then Bob will respond with B , giving Alice a reward of 2. So Alice should commit to X , in which case her reward would be 3.

(c) [10 points] Suppose Alice can commit first and decides to commit to a *mixed* strategy. Which mixed strategy would be best for her to commit to? What would be her reward in the resulting outcome? Justify your answer. Assume that Bob breaks ties in favor of Alice, that is, whenever Bob is indifferent between his two actions, he chooses the one that would give Alice a higher payoff.

[HINT: Find the highest payoff that Alice can obtain when she commits to a mixed strategy that makes Bob play A . Then find the highest payoff that Alice can obtain when she commits to a mixed strategy that makes Bob play B . And finally, choose the higher of the two payoffs.]

Solution: Suppose Alice commits to playing X w.p. p and Y w.p. $1 - p$, where $p \in [0, 1]$. We take two cases depending on what the response of Bob is.

- For Bob to respond with A , we must have $p > 1 - p \Leftrightarrow p > 1/2$, so $p \in (1/2, 1]$. In this case, Alice's reward is $3p + (1 - p) = 1 + 2p$, which is maximized at $p = 1$ in the range $p \in (1/2, 1]$, making Alice's reward $1 + 2 = 3$.
- For Bob to respond with B , we must have $p \leq 1 - p \Leftrightarrow p \leq 1/2$ (at $p = 1/2$, Bob breaks the tie by playing B), so $p \in [0, 1/2]$. In this case, Alice's reward is $5p + 2(1 - p) = 2 + 3p$, which is maximized at $p = 1/2$ in the range $p \in [0, 1/2]$, making Alice's reward $2 + 3 \cdot 1/2 = 7/2$.

Hence, the highest payoff that Alice can obtain is $7/2 = 3.5$, which is obtained when she commits to the mixed strategy of playing X and Y with probability $1/2$ each (and Bob responds with B).