

Q1. Nash Equilibria (25 pts)

Consider the following normal-form game involving two players. The row player (Alice) and the column player (Bob) have four actions each. In each cell of the payoff matrix, the first number indicates the payoff to the row player (Alice) and the second number indicates the payoff to the column player (Bob). Note that the payoffs are negative, so the players incur costs.

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
<i>W</i>	-4, -7	-6, -5	-5, -6	-6, -5
<i>X</i>	-5, -6	-1, -5	-2, -1	-5, -5
<i>Y</i>	-6, -6	-4, -5	-3, -3	-1, -2
<i>Z</i>	-7, -1	-3, -3	-6, -5	-7, -7

(a) [2.5 points] If Alice plays *Z*, what is Bob's best response? You do not need to justify your answer.

Solution: *A*

(b) [2.5 points] If Bob plays *C*, what is Alice's best response? You do not need to justify your answer.

Solution: *X*

(c) [5 points] It is possible to iteratively eliminate strictly dominated actions and reduce the game to a 2×2 game in which only two actions of each player remain. State which actions you would remove and in which order. You do not need to justify your answer.

Solution: *Z, A, W, B*

The reduced game is as follows.

	<i>C</i>	<i>D</i>
<i>X</i>	-2, -1	-5, -5
<i>Y</i>	-3, -3	-1, -2

(d) [5 points] In the reduced 2×2 game produced in part (c), find all (if any) pure Nash equilibria. You only need to identify the equilibria; no explanation is needed.

Solution: (*X, C*) and (*Y, D*)

(e) [10 points] In the reduced 2×2 game produced in part (c), find all mixed Nash equilibria, apart from those already found in part (d), using the indifference principle.

Solution: There are three remaining cases.

- *Bob does not randomize, Alice does.* Alice's rewards for her two actions are different for each action of Bob ($-2 \neq -3$ against *C*, and $-5 \neq -1$ against *D*). Hence, there are no NE in this case by the indifference principle.
- *Alice does not randomize, Bob does.* Similarly, Bob's rewards for his two actions are different for each action of Alice ($-1 \neq -5$ against *X*, and $-3 \neq -2$ against *Y*). Hence, there are no NE in this case either.
- *Alice and Bob both randomize.* Suppose Alice plays *X* w.p. p and *Y* w.p. $1 - p$, while Bob plays *C* w.p. q and *D* w.p. $1 - q$, where $p, q \in (0, 1)$. By the indifference principle for Alice, we need $-2q - 5(1 - q) = -3q - (1 - q) \Leftrightarrow 5q = 4 \Leftrightarrow q = 4/5$. By the indifference principle for Bob, we need $-p - 3(1 - p) = -5p - 2(1 - p) \Leftrightarrow 5p = 1 \Leftrightarrow p = 1/5$.

Hence, the only remaining equilibrium is the one in which Alice plays X w.p. $1/5$ and Y w.p. $4/5$, while Bob plays C w.p. $4/5$ and D w.p. $1/5$.

Q2. Pure Nash Equilibria (10 pts)

Consider a two-player game. A mixed strategy profile (s_1, s_2) consists of a mixed strategy s_1 of player 1 and a mixed strategy s_2 of player 2. Given the profile (s_1, s_2) , the payoffs of players 1 and 2 are denoted $u_1(s_1, s_2)$ and $u_2(s_1, s_2)$, respectively. Suppose (s_1^*, s_2^*) is a mixed Nash equilibrium of the game such that

$$\begin{aligned} u_1(s_1^*, s_2^*) &> u_1(s_1, s_2^*), \forall \text{ mixed strategies } s_1 \neq s_1^* \text{ of player 1,} \\ u_2(s_1^*, s_2^*) &> u_2(s_1^*, s_2), \forall \text{ mixed strategies } s_2 \neq s_2^* \text{ of player 2.} \end{aligned}$$

Either prove that (s_1^*, s_2^*) is necessarily a pure Nash equilibrium, or provide an example in which the above conditions are satisfied but (s_1^*, s_2^*) is not a pure Nash equilibrium.

NOTE: Recall that a pure strategy is also a mixed strategy by definition, and similarly, a pure Nash equilibrium is also a mixed Nash equilibrium by definition.

Solution: We prove that (s_1^*, s_2^*) is necessarily a pure Nash equilibrium.

Suppose for contradiction that s_1^* is not a pure strategy. Then at least two distinct actions a and a' of player 1 lie in the support of s_1^* . By the indifference principle, every action in the support of s_1^* is a best response to s_2^* . Hence, $u_1(a, s_2^*) = u_1(s_1^*, s_2^*)$. However, the pure strategy a is a mixed strategy different from s_1^* (because s_1^* plays a' with positive probability). So the first given condition implies $u_1(s_1^*, s_2^*) > u_1(a, s_2^*)$, which is a contradiction. Hence, s_1^* is a pure strategy.

By a symmetric argument using the second given condition, s_2^* is a pure strategy as well. Since (s_1^*, s_2^*) is a mixed Nash equilibrium in which both strategies are pure, it is a pure Nash equilibrium.

Q3. Stackelberg Equilibria (20 pts)

Consider the following game between two players. Again, the row player is Alice, the column player is Bob, the first number in each cell denotes the payoff to Alice, and the second number in each cell denotes the payoff to Bob.

	A	B
X	5, 4	0, 8
Y	0, 1	3, 2

(a) [5 points] Suppose both players have to choose their strategies simultaneously. Identify the unique Nash equilibrium of the game by iteratively eliminating strictly dominated actions. You only need to write down the actions removed, the order in which they are removed, and the unique Nash equilibrium; no further explanation is needed.

Solution: Actions removed in order: A, X . Nash equilibrium: (Y, B) .

(b) [5 points] If Bob can commit to a *pure* strategy and let Alice know of his strategy, what is the best pure strategy for him to commit to? What would be his reward in the resulting outcome? Explain your answer.

Solution: If Bob commits to A , then Alice will respond with X , giving Bob a reward of 4. If Bob commits to B , then Alice will respond with Y , giving Bob a reward of 2. So Bob should commit to A , in which case his reward would be 4.

(c) [10 points] If Bob can commit to a *mixed* strategy and let Alice know of his strategy, what is the best mixed strategy for him to commit to? What would be his reward in the resulting outcome? Explain your answer. If Alice is indifferent between her two pure strategies, assume that she plays the pure strategy that gives Bob a higher payoff.

HINT: You do not need to solve a complicated linear program for this question. Just think about which action Bob would want Alice to respond with.

Solution: Suppose Bob commits to playing A w.p. q and B w.p. $1 - q$, where $q \in [0, 1]$. Alice's reward is $5q$ from playing X and $3(1 - q)$ from playing Y . We take two cases depending on what the response of Alice is.

- *For Alice to respond with X* , we must have $5q \geq 3(1 - q) \Leftrightarrow q \geq 3/8$ (at $q = 3/8$, Alice is indifferent and breaks the tie in favor of Bob by responding with X , which gives Bob $13/2$ rather than $13/8$), so $q \in [3/8, 1]$. In this case, Bob's reward is $4q + 8(1 - q) = 8 - 4q$, which is maximized at $q = 3/8$ in the range $q \in [3/8, 1]$, making Bob's reward $8 - 4 \cdot 3/8 = 13/2$.
- *For Alice to respond with Y* , we must have $5q < 3(1 - q) \Leftrightarrow q < 3/8$, so $q \in [0, 3/8)$. In this case, Bob's reward is $q + 2(1 - q) = 2 - q \leq 2$.

Hence, the highest reward that Bob can obtain is $13/2 = 6.5$, which is obtained when he commits to the mixed strategy of playing A w.p. $3/8$ and B w.p. $5/8$ (and Alice responds with X).