

CSC304

Algorithmic Game Theory & Mechanism Design

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Announcements

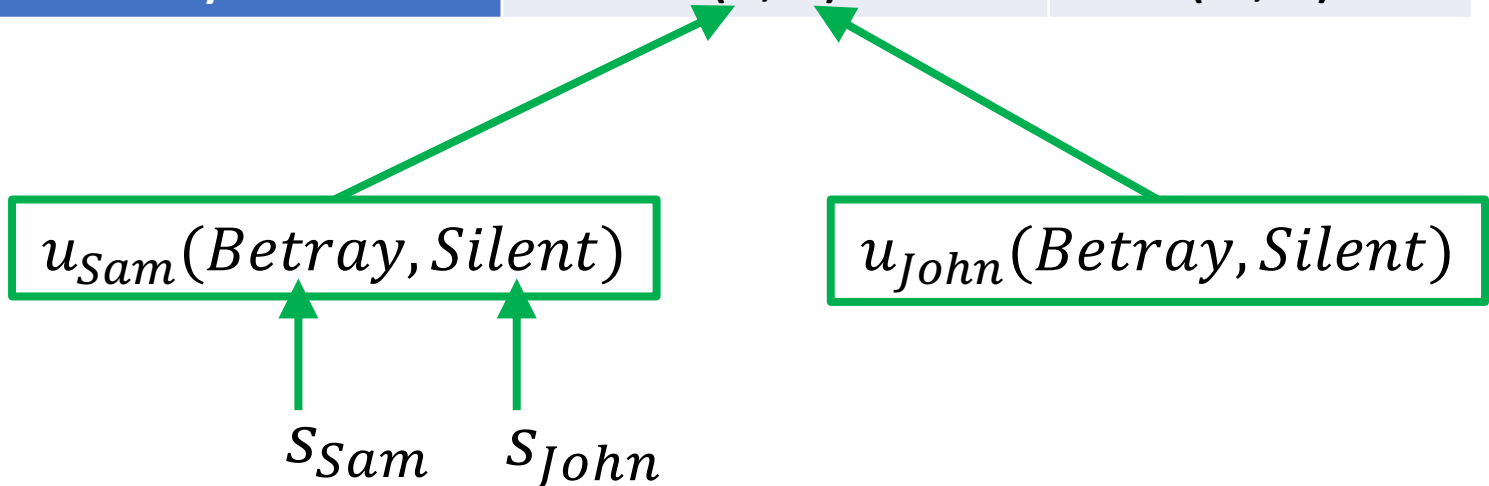
- Office hour slot
 - Mon, 3-4pm ET, starts next week

Recap: Normal Form Games

Recall: Prisoner's dilemma

$$S = \{\text{Silent}, \text{Betray}\}$$

		John's Actions	
		Stay Silent	Betray
Sam's Actions	Stay Silent	$(-1, -1)$	$(-3, 0)$
	Betray	$(0, -3)$	$(-2, -2)$



Recap: Domination

- Pure strategy s_i dominates pure strategy s'_i if player i is always “better off” playing s_i than s'_i , regardless of the strategies of other players.
- Two variants: weak and strict domination
 - $u_i(s_i, \vec{s}_{-i}) \geq u_i(s'_i, \vec{s}_{-i}), \forall \vec{s}_{-i}$ (needed for both)
 - Strict inequality for **some** \vec{s}_{-i} ← s_i *weakly dominates* s'_i
 - Strict inequality for **all** \vec{s}_{-i} ← s_i *strictly dominates* s'_i

Recap: Dominant Strategies

- (Pure) strategy s_i is a strictly (weakly) dominant strategy for player i if it strictly (weakly) dominates **every other (pure) strategy**
- Strict dominance is a strong concept
 - A player who has a strictly dominant strategy has no reason *not* to play it
 - If every player has a strictly dominant strategy, such strategies will very likely dictate the outcome of the game

Recap: Prisoner's Dilemma

		John's Actions	
		Stay Silent	Betray
Sam's Actions	Stay Silent	$(-1, -1)$	$(-3, 0)$
	Betray	$(0, -3)$	$(-2, -2)$

- Betraying is a strictly dominant strategy for each player

Iterated Elimination

- What if there are no dominant strategies?
 - No single strategy dominates every other strategy
 - But some strategies might still be dominated
- Assuming everyone knows everyone is rational...
 - Can remove their dominated strategies
 - Might reveal a newly dominant strategy
- Two variants depending on what we eliminate:
 - Only strictly dominated? Or also weakly dominated?

Iterated Elimination

- Toy example:
 - Microsoft vs Startup
 - Enter the market or stay out?

	Startup	
Microsoft		
Enter	(2, -2)	(4, 0)
Stay Out	(0, 4)	(0, 0)

- Q: Is there a dominant strategy for startup?
- Q: Do you see a rational outcome of the game?

Iterated Elimination

- More serious: “Guess $2/3$ of average”
 - Each student guesses a real number between 0 and 100 (inclusive)
 - The student whose number is the closest to $2/3$ of the average of all numbers wins!
- In-class poll!
- Recall: We have a unique optimal strategy only if everyone is rational, and everyone thinks everyone is rational, and so on.

Nash Equilibrium

- What if we don't find a unique outcome after iterated elimination of dominated strategies?

		Professor	
		Attend	Be Absent
Students	Attend	(3 , 1)	(-1 , -3)
	Be Absent	(-1 , -1)	(0 , 0)

Nash Equilibrium

- **Nash Equilibrium**

- A strategy profile \vec{s} is in Nash equilibrium if s_i is the best action for player i given that other players are playing \vec{s}_{-i}

$$u_i(s_i, \vec{s}_{-i}) \geq u_i(s'_i, \vec{s}_{-i}), \forall s'_i.$$



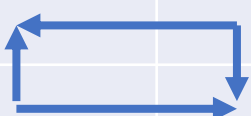
- Each player's strategy is only best *given* the strategies of others, and not *regardless*.
- You can't reason about a single player in isolation. You can only say whether you're in a NE after seeing the entire strategy profile.

Pure vs Mixed Nash Equilibria

- A **pure strategy** s_i is **deterministic**
 - That is, player i plays a single action w.p. 1
- A **mixed strategy** s_i can *possibly* randomize over actions
 - In a **fully-mixed strategy**, every action is played with a positive probability
- A strategy profile \vec{s} is pure if each s_i is pure
 - These are the “cells” in the normal form representation
- A **pure Nash equilibrium (PNE)** is a pure strategy profile that is a Nash equilibrium

Recap: Attend or Not

		Professor	
		Attend	Be Absent
Students	Attend	(3, 1)	(-1, -3)
	Be Absent	(-1, -1)	(0, 0)



- Pure Nash equilibria?

Pure Nash Equilibria

- **Best response**

- The best response of player i to others' strategies \vec{s}_{-i} is the highest reward action:

$$s_i^* \in \operatorname{argmax}_{s_i} u_i(s_i, \vec{s}_{-i})$$

- **Best-response diagram:**

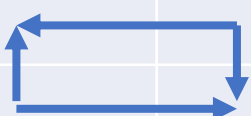
- From each cell \vec{s} , for each player i , draw an arrow to (s_i^*, \vec{s}_{-i}) , where $s_i^* = \text{player } i\text{'s best response to } \vec{s}_{-i}$
 - unless s_i is already a best response

- **Pure Nash equilibria (PNE)**

- Each player is already playing their best response
- **No outgoing arrows**

Example: Stag Hunt

		Hunter 1	
		Stag	Hare
Hunter 2	Stag	(4, 4)	(0, 2)
	Hare	(2, 0)	(1, 1)



- Game:
 - Each hunter decides to hunt stag or hare
 - Stag = 8 days of food, hare = 2 days of food
 - Catching stag requires both hunters, catching hare requires only one
 - If they catch one animal together, they share

- Pure Nash equilibria?

Recap: Prisoner's Dilemma

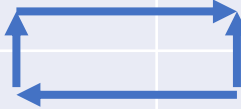
		John's Actions	
		Stay Silent	Betray
Sam's Actions	Stay Silent	$(-1, -1)$	$(-3, 0)$
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- Pure Nash equilibria?
- **Food for thought:**
 - What is the relation between iterated elimination of weakly/strictly dominated strategies and Nash equilibria?

Recap: Microsoft vs Startup

		Startup	
		Enter	Stay Out
Microsoft	Enter	(2, -2)	(4, 0)
	Stay Out	(0, 4)	(0, 0)



- Pure Nash equilibria?
- **Food for thought:**
 - What is the relation between iterated elimination of weakly/strictly dominated strategies and Nash equilibria?

Example Games

- Rock-Paper-Scissor : No PNE! **Why?**

P1 \ P2	Rock	Paper	Scissor
Rock	(0 , 0)	(-1 , 1)	(1 , -1)
Paper	(1 , -1)	(0 , 0)	(-1 , 1)
Scissor	(-1 , 1)	(1 , -1)	(0 , 0)

Nash's Beautiful Result

- **Nash's Theorem:**

- Every normal form game has **at least one (possibly mixed) Nash equilibrium.**
- Proof? We'll prove a special case later.

- We identify pure NE using best-response diagrams.

- How do we find mixed NE?

- **The Indifference Principle**

- *If (s_i, \vec{s}_{-i}) is a Nash equilibrium, then any action to which s_i assigns a positive probability must be a best action given \vec{s}_{-i} .*

*For each action a_i of player i satisfying $\Pr_{s_i}[a_i] > 0$:
 $u_i(a_i, \vec{s}_{-i}) \geq u_i(a'_i, \vec{s}_{-i})$ for all actions a'_i of player i .*

Revisiting Stag-Hunt

		Hunter 2	
		Stag	Hare
Hunter 1	Stag	(4, 4)	(0, 2)
	Hare	(2, 0)	(1, 1)

- Let's solve for symmetric mixed NE

- $s_1 = s_2 = (\text{Stag w.p. } p, \text{ Hare w.p. } 1 - p)$, where $p \in (0,1)$

- Indifference principle:

- Each player must be receiving equal reward from stag and hare given the other player's mixed strategy

- $\mathbb{E}[\text{Stag}] = p * 4 + (1 - p) * 0$

- $\mathbb{E}[\text{Hare}] = p * 2 + (1 - p) * 1$

- $4p = 2p + (1 - p) \Rightarrow p = 1/3$