CSC304
Algorithmic Game Theory & Mechanism Design

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Announcements

- Office hour slot
  - Mon, 3-4pm ET, starts next week

Recap: Normal Form Games

Recall: Prisoner's dilemma $S = \{\text{Silent}, \text{Betray}\}$

Sam's Actions \ John's Actions	Stay Silent	Betray
Stay Silent	(-1 , -1)	(-3 , 0)
Betray	(0 , -3)	(-2 , -2)

$u_{Sam}(Betray, Silent)$ $u_{John}(Betray, Silent)$

s_{Sam} s_{John}

Recap: Domination

- Pure strategy s_i dominates pure strategy s_i' if player i is always "better off" playing s_i than s_i', regardless of the strategies of other players.

- Two variants: weak and strict domination
  - $u_i(s_i, \vec{s}_{-i}) \geq u_i(s_i', \vec{s}_{-i}), \forall \vec{s}_{-i}$ (needed for both)
  - Strict inequality for some $\vec{s}_{-i}$ $\leftarrow$ s_i *weakly dominates* s_i'
  - Strict inequality for all $\vec{s}_{-i}$ $\leftarrow$ s_i *strictly dominates* s_i'

Recap: Dominant Strategies

- (Pure) strategy s_i is a strictly (weakly) dominant strategy for player i if it strictly (weakly) dominates every other (pure) strategy

- Strict dominance is a strong concept
  - A player who has a strictly dominant strategy has no reason *not* to play it
  - If every player has a strictly dominant strategy, such strategies will very likely dictate the outcome of the game

Recap: Prisoner's Dilemma

John's Actions / Sam's Actions	Stay Silent	Betray
Stay Silent	(-1 , -1)	(-3 , 0)
Betray	(0 , -3)	(-2 , -2)

- Betraying is a strictly dominant strategy for each player

Iterated Elimination

- What if there are no dominant strategies?
  - No single strategy dominates every other strategy
  - But some strategies might still be dominated

- Assuming everyone knows everyone is rational…
  - Can remove their dominated strategies
  - Might reveal a newly dominant strategy

- Two variants depending on what we eliminate:
  - Only strictly dominated? Or also weakly dominated?

Iterated Elimination

- Toy example:
  - Microsoft vs Startup
  - Enter the market or stay out?

Microsoft \ Startup	Enter	Stay Out
Enter	(2 , -2)	(4 , 0)
Stay Out	(0 , 4)	(0 , 0)

- Q: Is there a dominant strategy for startup?
- Q: Do you see a rational outcome of the game?

Iterated Elimination

- More serious: "Guess 2/3 of average"
  - Each student guesses a real number between 0 and 100 (inclusive)
  - The student whose number is the closest to 2/3 of the average of all numbers wins!

- In-class poll!

- Recall: We have a unique optimal strategy only if everyone is rational, and everyone thinks everyone is rational, and so on.

Nash Equilibrium

- What if we don't find a unique outcome after iterated elimination of dominated strategies?

Students \ Professor	Attend	Be Absent
Attend	**(3 , 1)**	**(-1 , -3)**
Be Absent	**(-1 , -1)**	**(0 , 0)**

Nash Equilibrium

- Nash Equilibrium
  - A strategy profile $\vec{s}$ is in Nash equilibrium if s_i is the best action for player i given that other players are playing $\vec{s}_{-i}$

$$u_i(s_i, \vec{s}_{-i}) \geq u_i(s_i', \vec{s}_{-i}), \forall s_i'$$

No quantifier on $\vec{s}_{-i}$

  - Each player's strategy is only best *given* the strategies of others, and not *regardless*.
  - You can't reason about a single player in isolation. You can only say whether you're in a NE after seeing the entire strategy profile.

Pure vs Mixed Nash Equilibria

- A pure strategy s_i is deterministic
  - That is, player i plays a single action w.p. 1
- A mixed strategy s_i can *possibly* randomize over actions
  - In a fully-mixed strategy, every action is played with a positive probability
- A strategy profile $\vec{s}$ is pure if each s_i is pure
  - These are the "cells" in the normal form representation
- A pure Nash equilibrium (PNE) is a pure strategy profile that is a Nash equilibrium

Recap: Attend or Not

Students \ Professor	Attend	Be Absent
Attend	(3 , 1)	(-1 , -3)
Be Absent	(-1 , -1)	(0 , 0)

- Pure Nash equilibria?

Pure Nash Equilibria

- Best response
  - The best response of player i to others' strategies $\vec{s}_{-i}$ is the highest reward action:

$$s_i^* \in \text{argmax}_{s_i} u_i(s_i, \vec{s}_{-i})$$

- Best-response diagram:
  - From each cell $\vec{s}$, for each player i, draw an arrow to $(s_i^*, \vec{s}_{-i})$, where s_i^* = player i's best response to $\vec{s}_{-i}$
    - unless s_i is already a best response

- Pure Nash equilibria (PNE)
  - Each player is already playing their best response
  - No outgoing arrows

Example: Stag Hunt

Hunter 1 / Hunter 2	Stag	Hare
Stag	(4 , 4)	(0 , 2)
Hare	(2 , 0)	(1 , 1)

- Game:
  - Each hunter decides to hunt stag or hare
  - Stag = 8 days of food, hare = 2 days of food
  - Catching stag requires both hunters, catching hare requires only one
  - If they catch one animal together, they share

- Pure Nash equilibria?

Recap: Prisoner's Dilemma

John's Actions / Sam's Actions	Stay Silent	Betray
Stay Silent	(-1 , -1)	(-3 , 0)
Betray	(0 , -3)	(-2 , -2)

- Pure Nash equilibria?

- Food for thought:
  - What is the relation between iterated elimination of weakly/strictly dominated strategies and Nash equilibria?

Recap: Microsoft vs Startup

Microsoft \ Startup	Enter	Stay Out
Enter	**(2 , -2)**	**(4 , 0)**
Stay Out	**(0 , 4)**	**(0 , 0)**

- Pure Nash equilibria?

- Food for thought:
  - What is the relation between iterated elimination of weakly/strictly dominated strategies and Nash equilibria?

Example Games

- Rock-Paper-Scissor : No PNE! Why?

P1 \ P2	Rock	Paper	Scissor
Rock	**(0 , 0)**	**(-1 , 1)**	**(1 , -1)**
Paper	**(1 , -1)**	**(0 , 0)**	**(-1 , 1)**
Scissor	**(-1 , 1)**	**(1 , -1)**	**(0 , 0)**

Nash's Beautiful Result

- Nash's Theorem:
  - Every normal form game has at least one (possibly mixed) Nash equilibrium.
  - Proof? We'll prove a special case later.

- We identify pure NE using best-response diagrams.
  - How do we find mixed NE?

- The Indifference Principle
  - *If $(s_i, \vec{s}_{-i})$ is a Nash equilibrium, then any action to which s_i assigns a positive probability must be a best action given $\vec{s}_{-i}$.*

For each action a_i of player i satisfying $\Pr_{s_i}[a_i] > 0$:
$u_i(a_i, \vec{s}_{-i}) \geq u_i(a_i', \vec{s}_{-i})$ *for all actions a_i' of player i.*

Revisiting Stag-Hunt

Hunter 1 \ Hunter 2	Stag	Hare
Stag	(4 , 4)	(0 , 2)
Hare	(2 , 0)	(1 , 1)

- Let's solve for symmetric mixed NE
  - $s_1 = s_2 =$ (Stag w.p. p, Hare w.p. $1 - p$), where $p \in (0,1)$
- Indifference principle:
  - Each player must be receiving equal reward from stag and hare given the other player's mixed strategy
  - $\mathbb{E}[\text{Stag}] = p * 4 + (1-p) * 0$
  - $\mathbb{E}[\text{Hare}] = p * 2 + (1-p) * 1$
  - $4p = 2p + (1-p) \;\; \Rightarrow \;\; p = 1/3$