CSC2556

Lecture 8

Fair Division 3: Rent Division

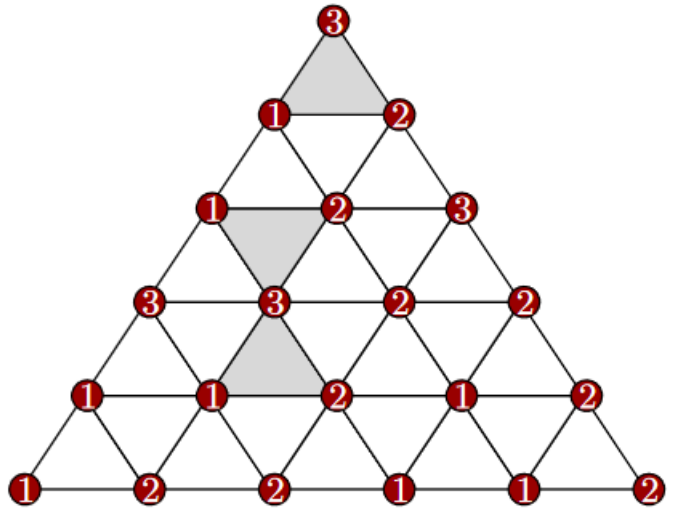
Rent Division

- An apartment with n roommates & n rooms
- Roommates have preferences over the rooms
- Total rent is R

- Goal: Find an allocation of rooms to roommates & a division of the total rent that is envy-free.

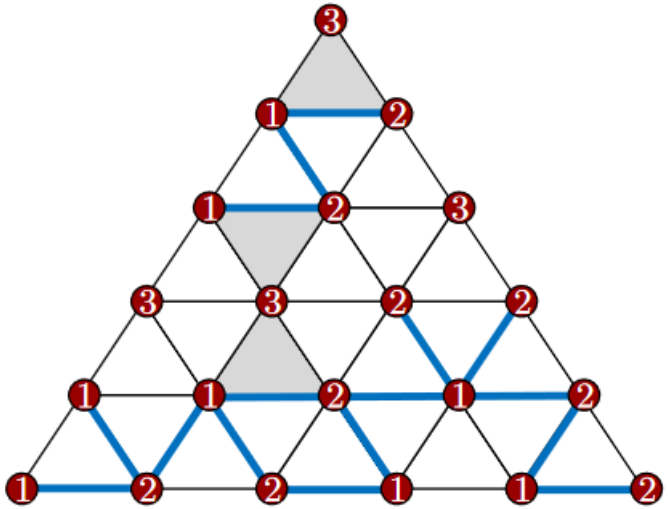
Sperner's Lemma

- Triangle T partitioned into elementary triangles
- Sperner Labeling:
  - Label vertices $\{1,2,3\}$
  - Main vertices are different
  - Vertices between main vertices i and j are each labeled i or j
- Lemma:
  - Any Sperner labeling contains at least one "fully labeled" (1-2-3) elementary triangle.



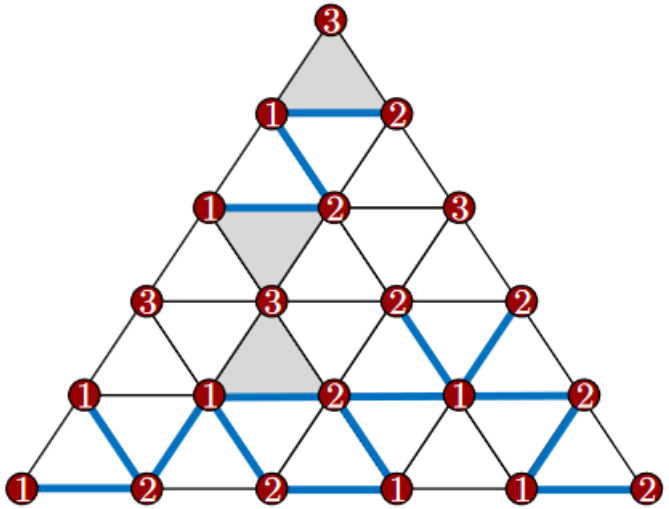
Sperner's Lemma

- Doors: 1-2 edges
- Rooms: elementary triangles
- Claim: #doors on the boundary of T is odd
- Claim: A fully labeled (123) room has 1 door. Every other room has 0 or 2 doors.



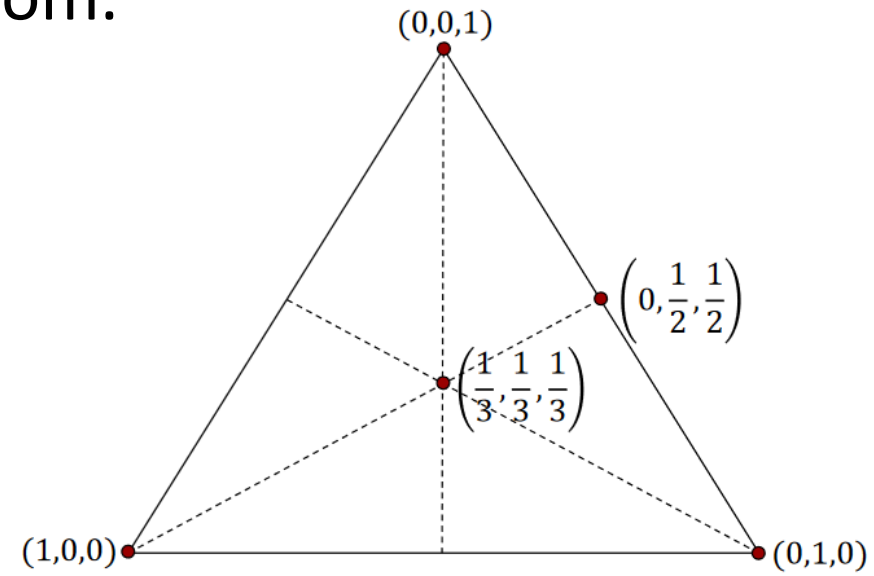
Sperner's Lemma

- Start at a door on boundary, and walk through it
- Either found a fully labeled room, or it has another door
- No room visited twice
- Eventually, find a fully labeled room or back out through another door on boundary
- But #doors on boundary is odd. ■



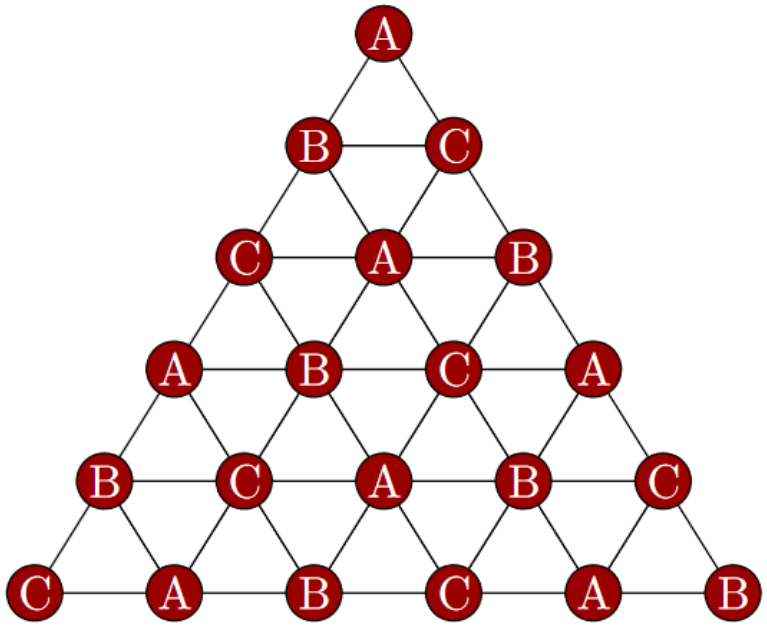
Fair Rent Division

- Three housemates A, B, C
- Goal: Divide total rent between three rooms so that at those rents, each person wants a different room.
- Without loss of generality, say the total rent is 1.
  - Represent possible partitions of rent as a triangle.



Fair Rent Division

- "Triangulate" and assign "ownership" of each vertex to A, B, or C so that each elementary triangle is an ABC triangle



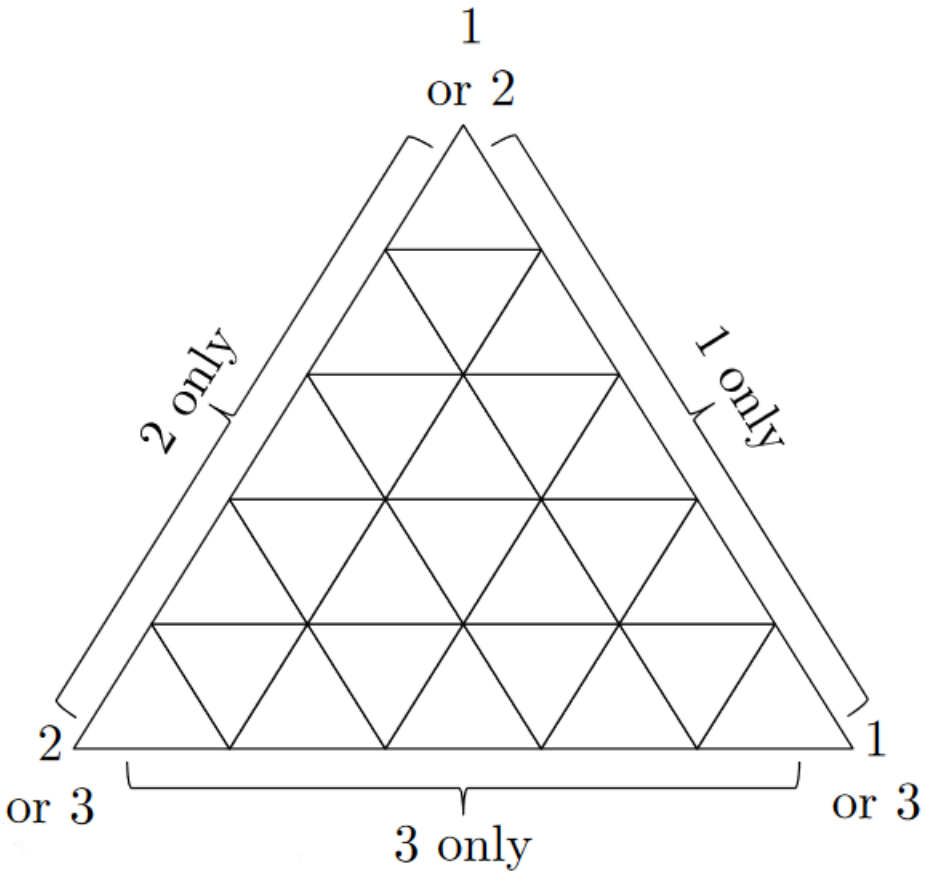
Fair Rent Division

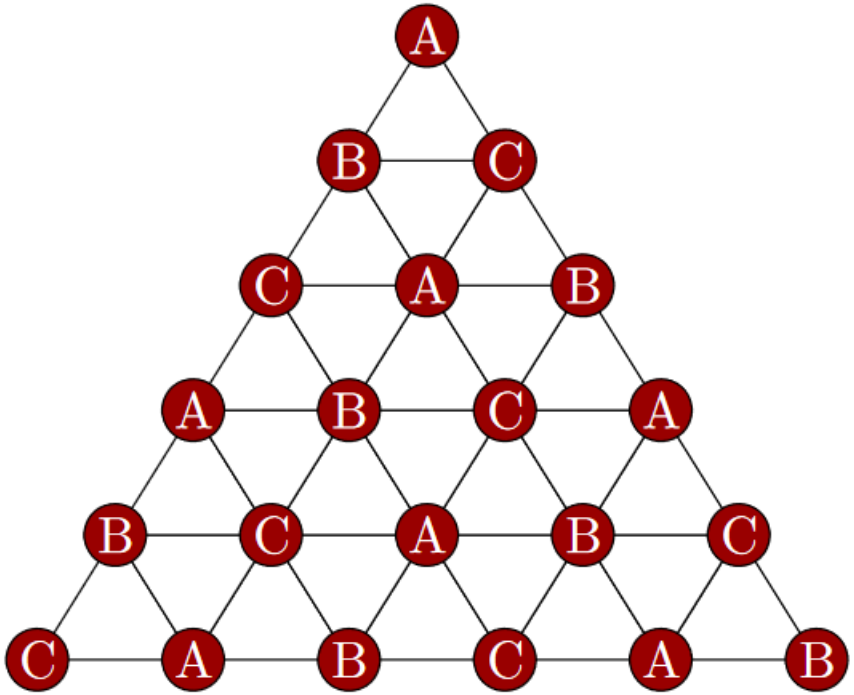
- Ask the owner of each vertex v:
  - Which room do you prefer if the rent division is given by the coordinates of v?

- Gives us a 1-2-3 labeling of the triangulation.
- Assumption: Each roommate prefers any free room over any paid room.
  - "Miserly roommates" assumption

Fair Rent Division

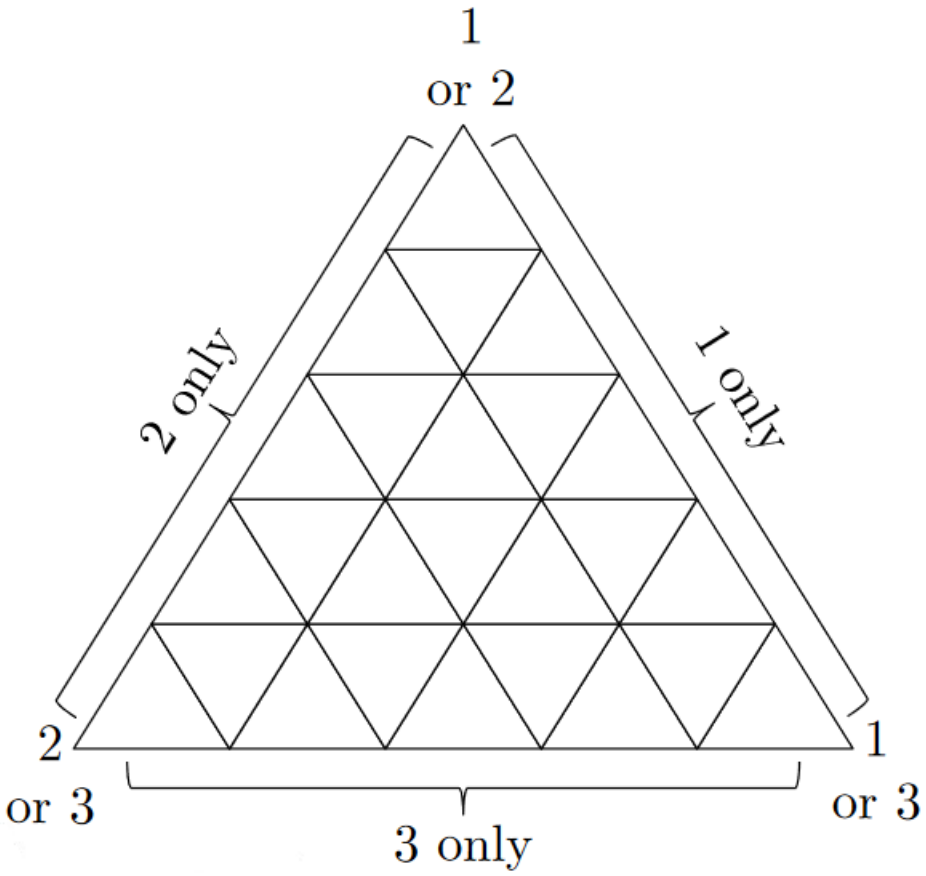
- This dictates the choice of rooms on the edges of T

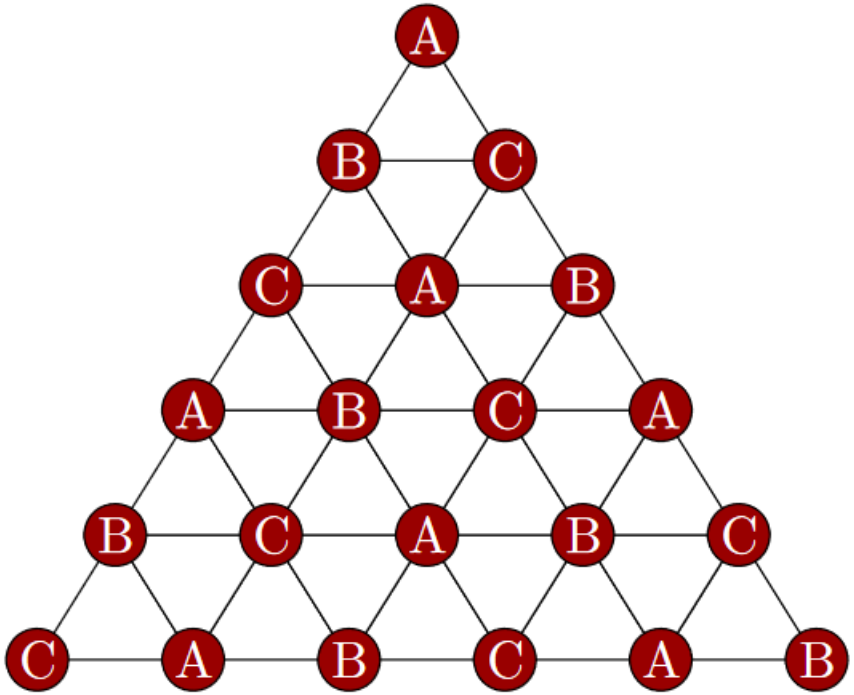




Fair Rent Division

- Sperner's Lemma: There must be a 1-2-3 triangle.





Fair Rent Division

- The three roommates prefer different rooms…
  - But at slightly different rent divisions.
  - Approximately envy-free.

- By making the triangulations finer, we can increase accuracy.
  - In the limit, we obtain an envy-free allocation.

- This technique generalizes to more roommates [Su 1999].

Quasi-Linear Utilities

- A different model:
  - Value of roommate i for room r = $v_{i,r}$
  - Rent for room r = p_r
  - Utility to agent i for getting room r = $v_{i,r} - p_r$

- We need to find an assignment A of rooms to roommates and a price vector p such that
  - Total rent: $R = \sum_r p_r$
  - Envy-freeness: $v_{i,A_i} - p_{A_i} \geq v_{i,A_j} - p_{A_j}$

Quasi-Linear Utilities

- Theorem: An envy-free (A, p) always exists!
  - We'll skip this proof.
- Theorem: If (A, p) is envy-free, $\sum_i v_{i,A_i}$ is maximized.
  - Implied by "1st fundamental theorem of welfare economics"
  - As a consequence, (A, p) is Pareto optimal.
  - Easy proof!
- Theorem: If (A, p) is envy-free and A' maximizes $\sum_i v_{i,A'_i}$ then (A', p) is envy-free.
  - Further, $v_{i,A_i} - p_{A_i} = v_{i,A'_i} - p_{A'_i}$ for every agent i
  - Implied by "2nd fundamental theorem of welfare economics"
  - Easy proof!

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Share Rent

Split Fare

Assign Credit

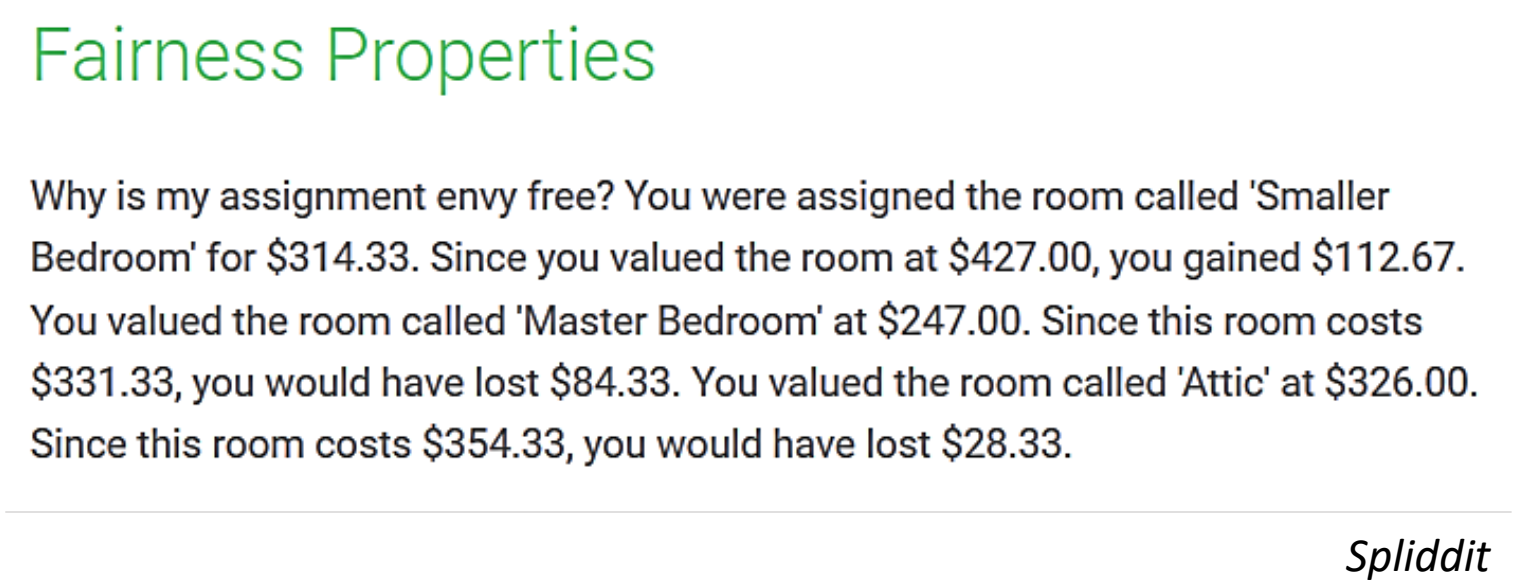
Divide Goods

Distribute Tasks

Suggest an App

Which Model Is Better?

- Advantage of quasi-linear utilities:
  - One-shot preference elicitation
    - Players directly report their values for the different rooms
  - Easy to explain the fairness guarantee



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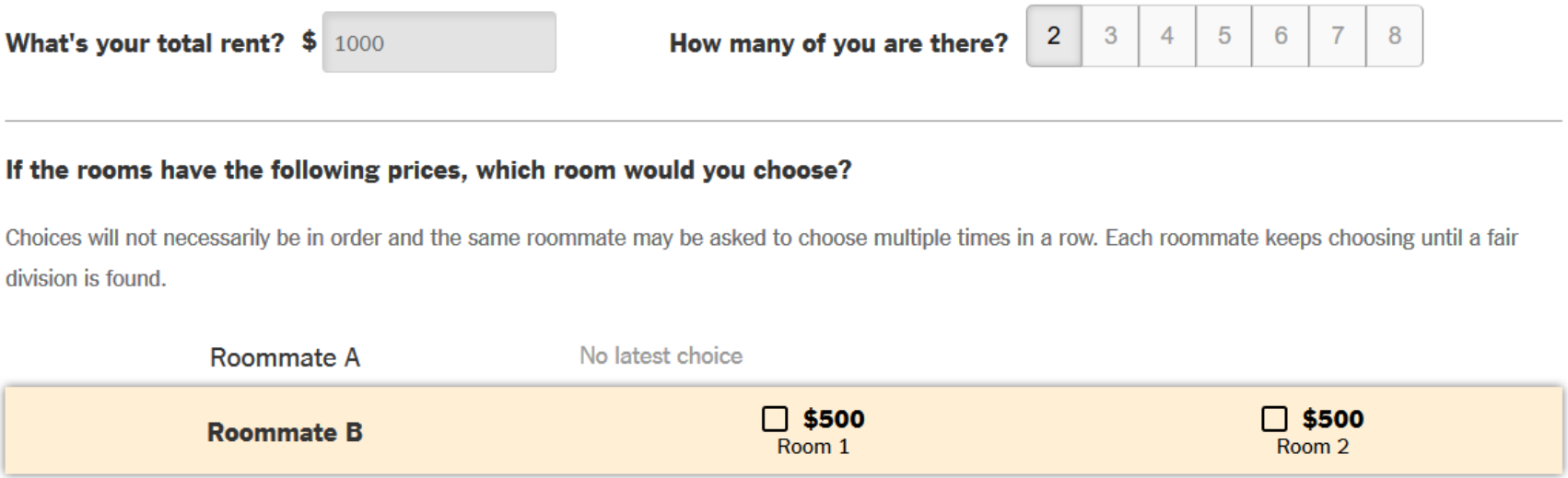
Which Model Is Better?

- Advantage of miserly roommates model:
  - Allows arbitrary preferences subject to a simple assumption
  - Easy queries: "Which room do you prefer at these prices?"



The New York Times