CSC2556

Lecture 5

Matching
- Stable Matching
- Kidney Exchange [Slides : Ariel D. Procaccia]

Announcements

- The assignment is up!
  - It is complete, and no more questions will be added.
  - After today's class, you will be able to attempt all but one question.
  - We will cover material for the last question in next class.
  - Due: March 11th by 11:59PM (~ 3 weeks from now)

- Project proposal
  - Due: March 1st by 12:59PM (i.e., before the class)
  - Will put up sample project ideas this weekend on Piazza.
  - If you have trouble finding a project idea, meet me.

Stable Matching

- Recap Graph Theory:

- In graph $G = (V, E)$, a matching $M \subseteq E$ is a set of edges with no common vertices
  - That is, each vertex should have at most one incident edge
  - A matching is perfect if no vertex is left unmatched.

- G is a bipartite graph if there exist V_1, V_2 such that $V = V_1 \cup V_2$ and $E \subseteq V_1 \times V_2$

Stable Marriage Problem

- Bipartite graph, two sides with equal vertices
  - n men and n women (old school terminology ☹)
- Each man has a ranking over women & vice versa
  - E.g., Eden might prefer Alice $\succ$ Tina $\succ$ Maya
  - And Tina might prefer Tony $\succ$ Alan $\succ$ Eden
- Want: a perfect, stable matching
  - Match each man to a unique woman such that no pair of man m and woman w prefer each other to their current matches (such a pair is called a "blocking pair")

Example: Preferences

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

 $\succ$  $\succ$ 

Example: Matching 1

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

Question: Is this a stable matching?

Example: Matching 1

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

No, Albert and Emily form a **blocking pair**.

Example: Matching 2

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

Question: How about this matching?

Example: Matching 2

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

Yes! (Charles and Fergie are unhappy, but helpless.)

Does a stable matching always exist in the marriage problem?

Can we compute it in a strategyproof way?

Can we compute it efficiently?

Gale-Shapley 1962

- Men-Proposing Deferred Acceptance (MPDA):

1. Initially, no one has proposed, no one is matched.
2. While some man m is unengaged:
   - $w \leftarrow m$'s most preferred woman to whom m has not proposed yet
   - m proposes to w
   - If w is unengaged:
     - m and w are engaged
   - Else if w prefers m to her current partner m'
     - m and w are engaged, m' becomes unengaged
   - Else: w rejects m
3. Match all engaged pairs.

Example: MPDA

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

= proposed = engaged = rejected

Running Time

- Theorem: DA terminates in polynomial time (at most n^2 iterations of the outer loop)

- Proof:
  - In each iteration, a man proposes to someone to whom he has never proposed before.
  - n men, n women $\rightarrow n \times n$ possible proposals
  - Can actually tighten a bit to $n(n-1)+1$ iterations

- At termination, it must return a perfect matching.

Stable Matching

- Theorem: DA always returns a stable matching.

- Proof by contradiction:
  - Assume (m, w) is a blocking pair.

  - Case 1: m never proposed to w
    - m cannot be unmatched o/w algorithm would not terminate.
    - Men propose in the order of preference.
    - Hence, m must be matched with a woman he prefers to w
    - (m, w) is not a blocking pair

Stable Matching

- Theorem: DA always returns a stable matching.

- Proof by contradiction:
  - Assume (m, w) is a blocking pair.

  - Case 2: m proposed to w
    - w must have rejected m at some point
    - Women only reject to get better partners
    - w must be matched at the end, with a partner she prefers to m
    - (m, w) is not a blocking pair

Men-Optimal Stable Matching

- The stable matching found by MPDA is special.
- Valid partner: For a man m, call a woman w a valid partner if (m, w) is in *some* stable matching.
- Best valid partner: For a man m, a woman w is the best valid partner if she is a valid partner, and m prefers her to every other valid partner.
  - Denote the best valid partner of m by $best(m)$.

Men-Optimal Stable Matching

- Theorem: Every execution of MPDA returns the "men-optimal" stable matching: every man is matched to his best valid partner.
  - Surprising that this is a matching. E.g., it means two men cannot have the same best valid partner!

- Theorem: Every execution of MPDA produces the "women-pessimal" stable matching: every woman is matched to her worst valid partner.

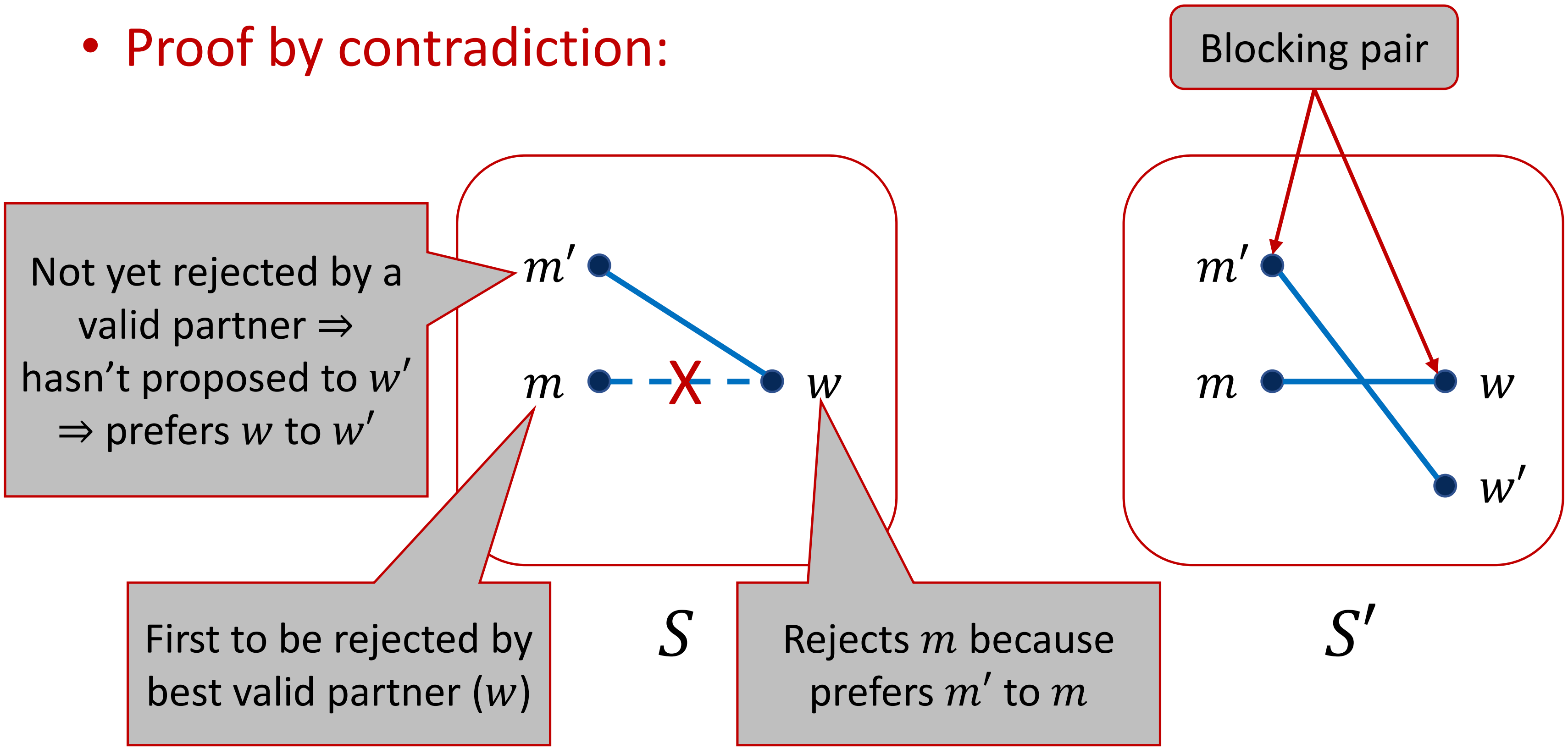
Men-Optimal Stable Matching

- Theorem: Every execution of MPDA returns the men-optimal stable matching.

- Proof by contradiction:
  - Let S = matching returned by MPDA.
  - $m \leftarrow$ first man rejected by $best(m) = w$
  - $m' \leftarrow$ the more preferred man due to which w rejected m
  - w is valid for m, so (m, w) part of stable matching S'
  - $w' \leftarrow$ woman m' is matched to in S'
  - We show that S' cannot be stable because (m', w) is a blocking pair.

Men-Optimal Stable Matching

- Theorem: Every execution of MPDA returns the men-optimal stable matching.
- Proof by contradiction:



Strategyproofness

- Theorem: MPDA is strategyproof for men.
  - We'll skip the proof of this.
  - Actually, it is group-strategyproof.

- But the women might gain by misreporting.

- Theorem: No algorithm for the stable matching problem is strategyproof for both men and women.

Women-Proposing Version

- Women-Proposing Deferred Acceptance (WPDA)
  - Just flip the roles of men and women
  - Strategyproof for women, not strategyproof for men
  - Returns the women-optimal and men-pessimal stable matching

Extensions

- Unacceptable matches
  - Allow every agent to report a partial ranking
  - If woman w does not include man m in her preference list, it means she would rather be unmatched than matched with m. And vice versa.
  - (m, w) is blocking if each prefers the other over their current state (matched with another partner or unmatched)
  - Just m (or just w) can also be blocking if they prefer being unmatched than be matched to their current partner

- Magically, DA still produces a stable matching.

Extensions

- Resident Matching (or College Admission)
  - Men → residents (or students)
  - Women → hospitals (or colleges)
  - Each side has a ranked preference over the other side
  - But each hospital (or college) q can accept $c_q > 1$ residents (or students)
  - Many-to-one matching

- An extension of Deferred Acceptance works
  - Resident-proposing (resp. hospital-proposing) results in resident-optimal (resp. hospital-optimal) stable matching

Extensions

- For ~20 years, most people thought that these problems are very similar to the stable marriage problem

- Roth [1985] shows:
  - No stable matching algorithm is strategyproof for hospitals (or colleges).

Extensions

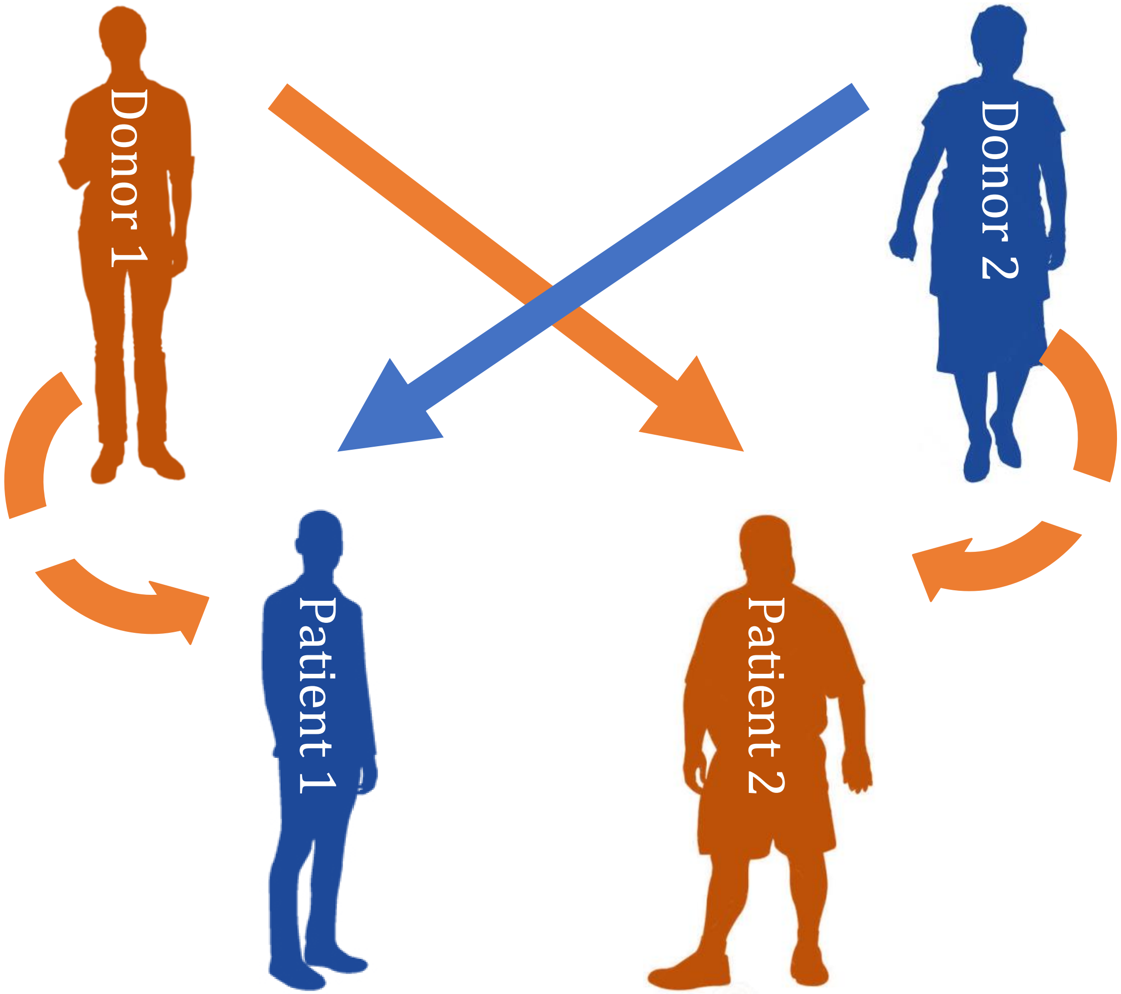
- Roommate Matching
  - Still one-to-one matching
  - But no partition into men and women
    - "Generalizing from bipartite graphs to general graphs"
  - Each of n agents submits a ranking over the other $n - 1$ agents

- Unfortunately, there are instances where no stable matching exist.
  - A variant of DA can still find a stable matching *if* it exists.
  - Due to Irving [1985]

NRMP: Matching in Practice

- 1940s: Decentralized resident-hospital matching
  - Markets “unralveled”, offers came earlier and earlier, quality of matches decreased
- 1950s: NRMP introduces centralized “clearinghouse”
- 1960s: Gale-Shapley introduce DA
- 1984: Al Roth studies NRMP algorithm, finds it is really a version of DA!
- 1970s: Couples increasingly don’t use NRMP
- 1998: NRMP implements matching with couple constraints (stable matchings may not exist anymore…)
- More recently, DA applied to college admissions

Kidney Exchange



Incentives

- A decade ago kidney exchanges were carried out by individual hospitals
- Today there are nationally organized exchanges; participating hospitals have little other interaction
- It was observed that hospitals match easy-to-match pairs internally, and enroll only hard-to-match pairs into larger exchanges
- Goal: incentivize hospitals to enroll all their pairs

The strategic model

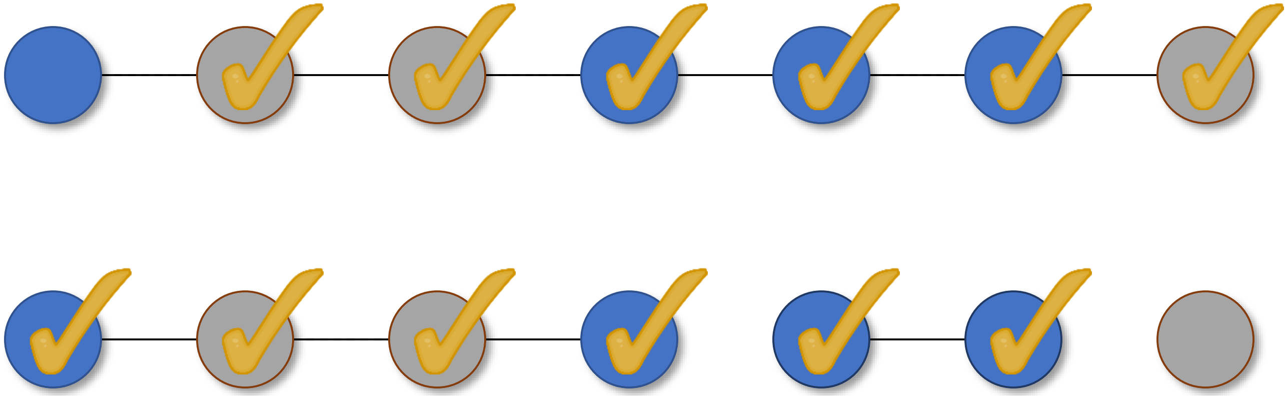
- Undirected graph, only pairwise matches
  - Vertex = donor-patient pair
  - Edge = compatibility

- Each agent controls a subset of vertices
  - Possible strategy: hide some vertices (match internally), and only reveal others
  - Utility of agent = # its matched vertices (self-matched + matched by mechanism)

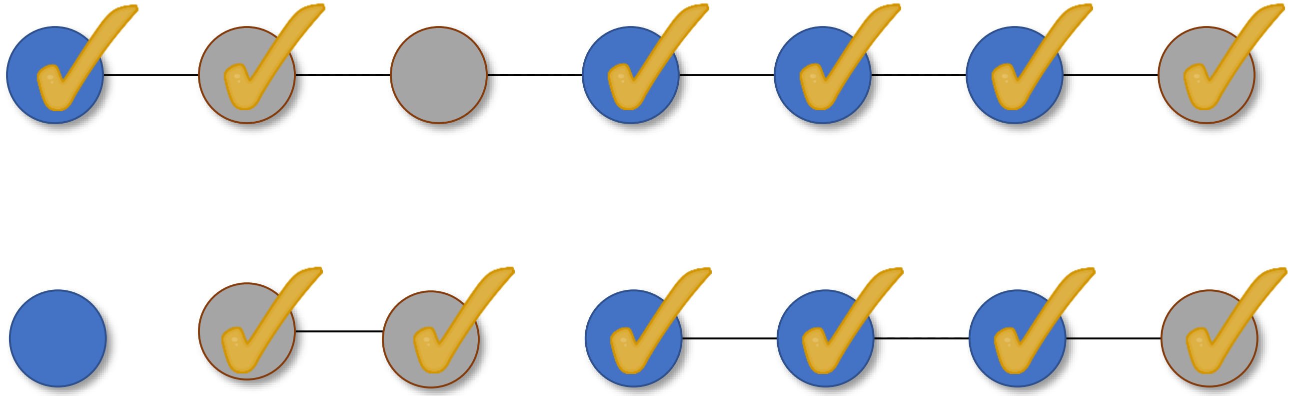
The strategic model

- Mechanism:
  - Input: revealed vertices by agents (edges are public)
  - Output: matching
- Target: # matched vertices
- Strategyproof (SP): If no agent benefits from hiding vertices irrespective of what other agents do.

OPT is manipulable

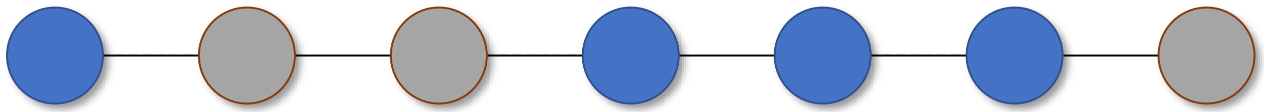


OPT is manipulable



Approximating SW

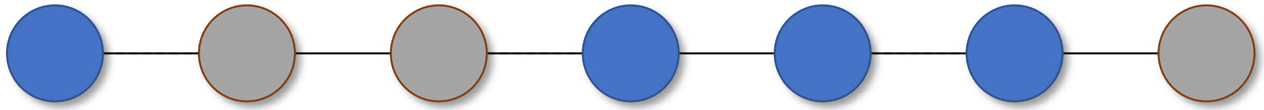
- Theorem [Ashlagi et al. 2010]: No deterministic SP mechanism can give a $2 - \epsilon$ approximation
- Proof:



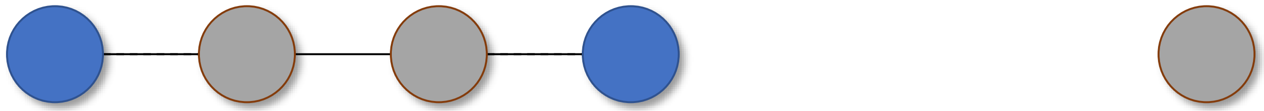
  - No perfect matching exists.
  - Any algorithm must match at most three blue nodes, or at most two gray nodes.

Approximating SW

- Theorem [Ashlagi et al. 2010]: No deterministic SP mechanism can give a $2-\epsilon$ approximation
- Proof:

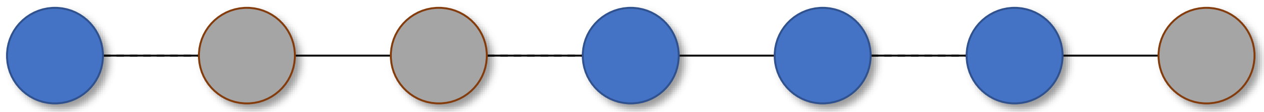


  - Suppose the algorithm matches at most three blue nodes
    - Cannot match both blue nodes in the following graph, otherwise blue agent has an incentive to hide nodes.
    - Must return a matching of size 1 when a matching of size 2 exists.

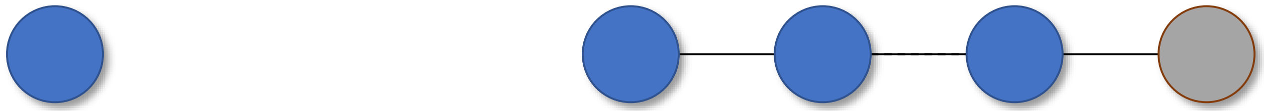


Approximating SW

- Theorem [Ashlagi et al. 2010]: No deterministic SP mechanism can give a $2 - \epsilon$ approximation
- Proof:



  - Suppose the algorithm matches at most two gray nodes
    - Cannot match the gray node in the following graph, otherwise the gray agent has an incentive to hide nodes.
    - Must return a matching of size 1 when a matching of size 2 exists.



Approximating SW

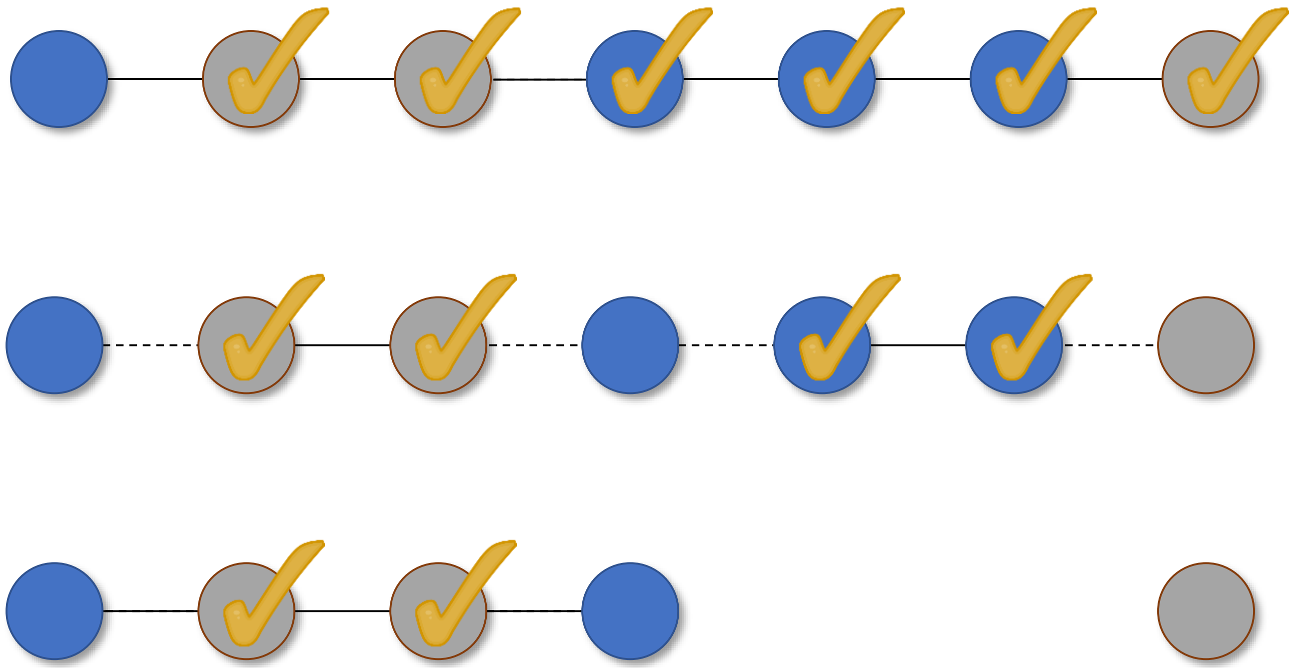
- Theorem [Kroer and Kurokawa 2013]: No randomized SP mechanism can give a $\frac{6}{5} - \epsilon$ approximation.
- Proof: Homework!

SP mechanism: Take 1

- Assume two agents

- $\text{M}\small{\text{ATCH}}_{\{\{1\},\{2\}\}}$ mechanism:
  - Consider matchings that maximize the number of "internal edges" for each agent.
  - Among these return, a matching with max overall cardinality.

Another example



Guarantees

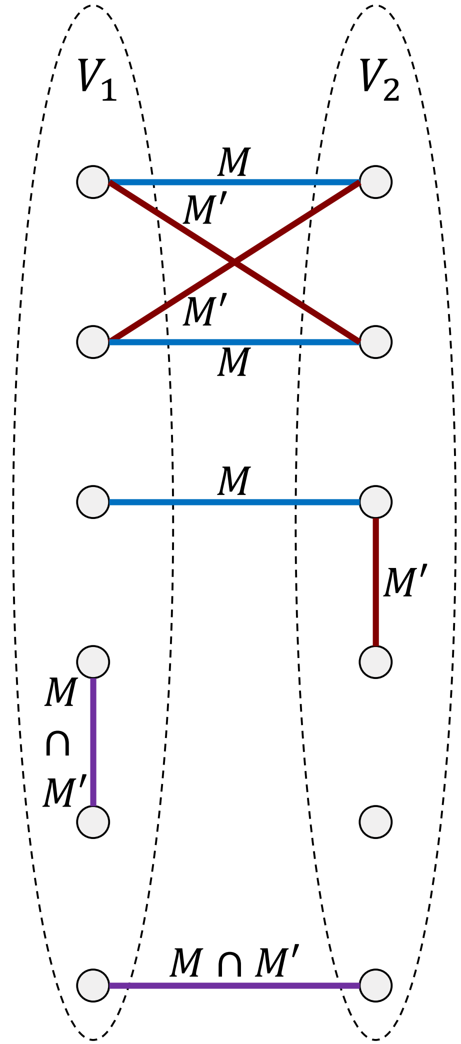
- $\text{M}\textsc{atch}_{\{\{1\},\{2\}\}}$ gives a 2-approximation
  - Cannot add more edges to matching
  - For each edge in optimal matching, one of the two vertices is in mechanism's matching

- Theorem (special case): $\text{M}\textsc{atch}_{\{\{1\},\{2\}\}}$ is strategyproof for two agents.

Proof

- M = matching when player 1 is honest, M' = matching when player 1 hides vertices
- $M \Delta M'$ consists of paths and even-length cycles, each consisting of alternating M, M' edges

What's wrong with the illustration on the right?



Proof

- Consider a path in $M \Delta M'$, denote its edges in M by P and its edges in M' by P'

- Consider sets P_{11}, P_{22}, P_{12} containing edges of P among V_1, among V_2, and between V_1- V_2
  - Same for $P'_{11}, P'_{22}, P'_{12}$

- Note that $|P_{11}| \geq |P'_{11}|$
  - Property of the algorithm

Proof

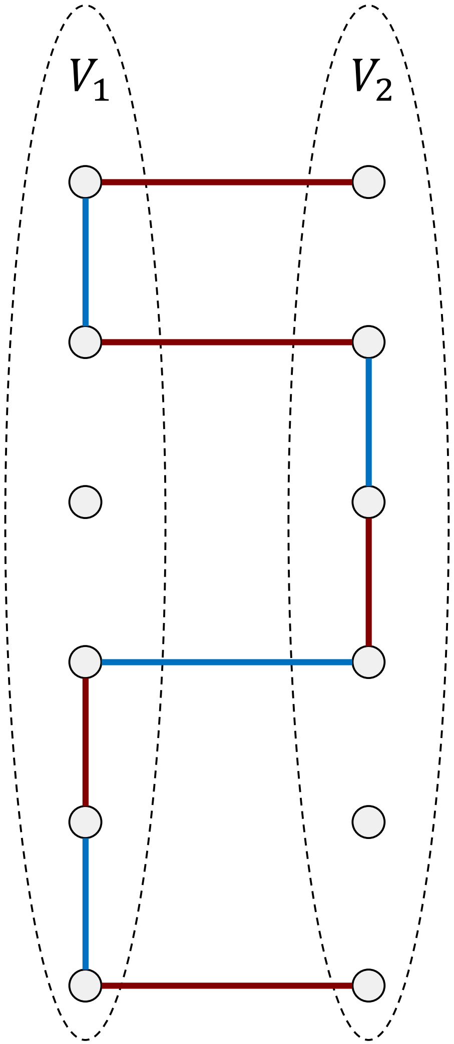
- Case 1: $|P_{11}| = |P'_{11}|$

- Agent 2's vertices don't change, so $|P_{22}| = |P'_{22}|$
- M is max cardinality $\Rightarrow |P_{12}| \geq |P'_{12}|$
- $U_1(P) = 2|P_{11}| + |P_{12}|$

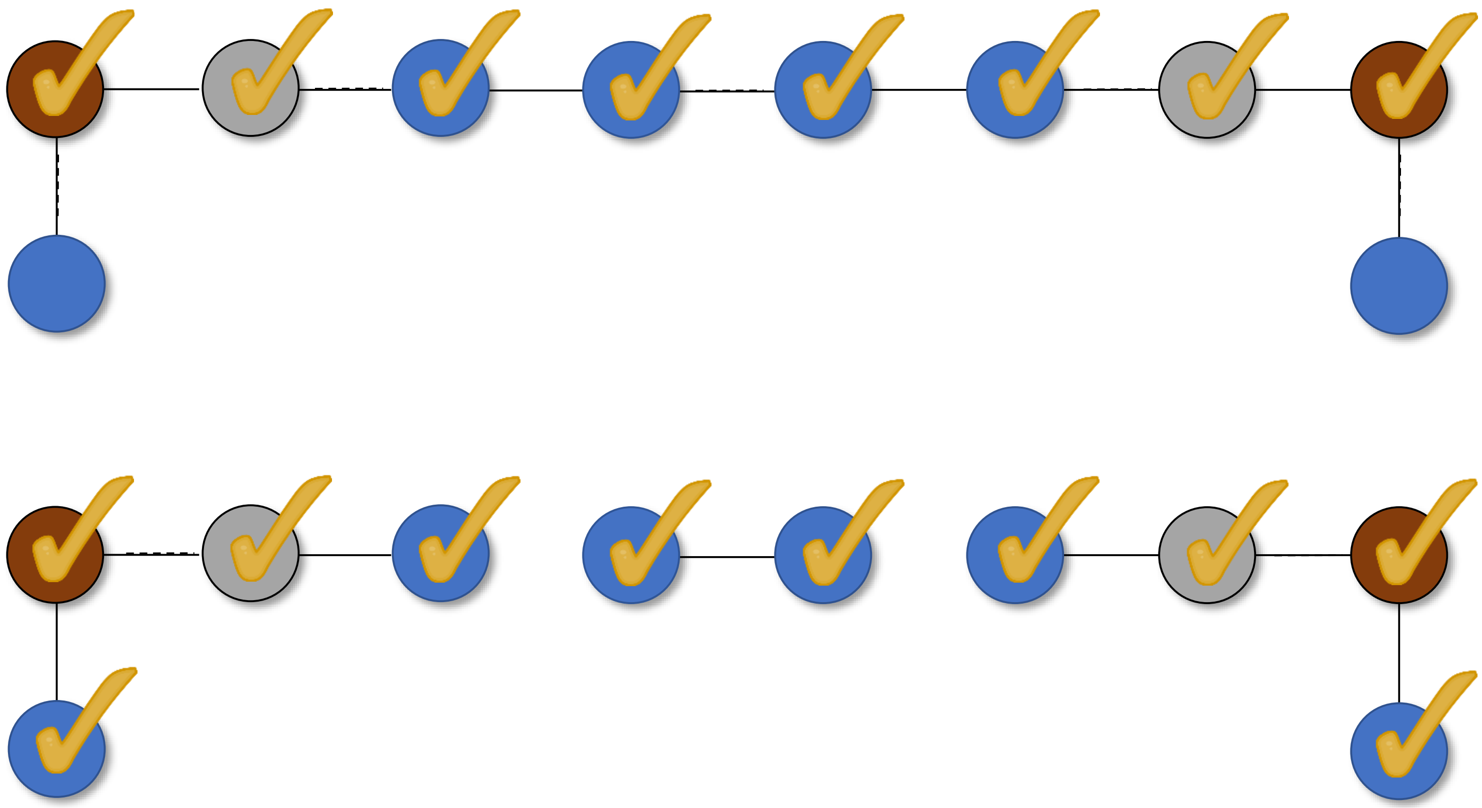
$$\geq 2|P'_{11}| + |P'_{12}| = U_1(P')$$

Proof

- Case 2: $|P_{11}| > |P'_{11}|$
- $|P_{12}| \geq |P'_{12}| - 2$
  - Every sub-path within V_2 is of even length
  - Pair up edges of P_{12} and P'_{12}, except maybe the first and the last
- $U_1(P) = 2|P_{11}| + |P_{12}|$
  $\geq 2(|P'_{11}| + 1) + |P'_{12}| - 2$
  $= U_1(P')$ ∎



The case of 3 players



SP Mechanism: Take 2

- Let $\Pi = (\Pi_1, \Pi_2)$ be a bipartition of the players

- MATCH_Π mechanism:
  - Consider matchings that maximize the number of "internal edges" and do not have any edges between different players on the same side of the partition
  - Among these return a matching with max cardinality (need tie breaking)

Eureka?

- Theorem [Ashlagi et al. 2010]: $\textsc{Match}_{\Pi}$ is strategyproof for any number of agents and any partition Π.
- Recall: For $n = 2$, $\textsc{Match}_{\{\{1\},\{2\}\}}$ is a 2-approximation
- Question: $n = 3$, $\textsc{Match}_{\{\{1\},\{2,3\}\}}$ approximation?

1. 2
2. 3
3. 4
4. More than 4

The Mechanism

- The MIX-AND-MATCH mechanism:
  - Mix: choose a random partition Π
  - Match: Execute MATCH$_{\Pi}$

- Theorem [Ashlagi et al. 2010]: MIX-AND-MATCH is strategyproof and a 2-approximation.

- We only prove the approximation ratio.

Proof

- $M^* =$ optimal matching
- Claim: I can create a matching M' such that
  - M' is max cardinality on each V_i, and
  - $\sum_i |M'_{ii}| + \frac{1}{2}\sum_{i \neq j} |M'_{ij}| \geq \sum_i |M^*_{ii}| + \frac{1}{2}\sum_{i \neq j} |M^*_{ij}|$

  - $M^{**} =$ max cardinality on each V_i
  - For each path P in $M^* \Delta M^{**}$, add $P \cap M^{**}$ to M' if M^{**} has more internal edges than M^*, otherwise add $P \cap M^*$ to M'
  - For every internal edge M' gains relative to M^*, it loses at most one edge overall ■

Proof

- Fix Π and let M^{Π} be the output of $\text{M}{\small\text{ATCH}}_{\Pi}$

- The mechanism returns max cardinality across Π subject to being max cardinality internally, therefore

$$\sum_{i} \left|M_{ii}^{\Pi}\right| + \sum_{i \in \Pi_1, j \in \Pi_2} \left|M_{ij}^{\Pi}\right| \geq \sum_{i} \left|M'_{ii}\right| + \sum_{i \in \Pi_1, j \in \Pi_2} \left|M'_{ij}\right|$$

Proof

$$\mathbb{E}\left[\left|M^{\Pi}\right|\right] = \frac{1}{2^n}\sum_{\Pi}\left(\sum_i \left|M_{ii}^{\Pi}\right| + \sum_{i\in\Pi_1, j\in\Pi_2}\left|M_{ij}^{\Pi}\right|\right)$$

$$\geq \frac{1}{2^n}\sum_{\Pi}\left(\sum_i \left|M'_{ii}\right| + \sum_{i\in\Pi_1, j\in\Pi_2}\left|M'_{ij}\right|\right)$$

$$= \sum_i \left|M'_{ii}\right| + \frac{1}{2^n}\sum_{\Pi}\sum_{i\in\Pi_1, j\in\Pi_2}\left|M'_{ij}\right|$$

$$= \sum_i \left|M'_{ii}\right| + \frac{1}{2}\sum_{i\neq j}\left|M'_{ij}\right| \geq \sum_i \left|M^*_{ii}\right| + \frac{1}{2}\sum_{i\neq j}\left|M^*_{ij}\right|$$

$$\geq \frac{1}{2}\sum_i \left|M^*_{ii}\right| + \frac{1}{2}\sum_{i\neq j}\left|M^*_{ij}\right| = \frac{1}{2}\left|M^*\right| \;\blacksquare$$