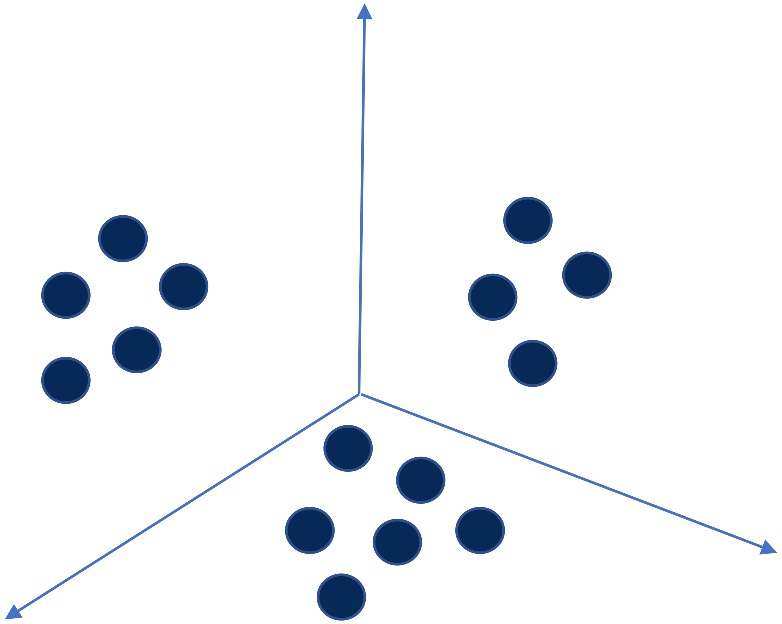
CSC2421
Fairness in Clustering

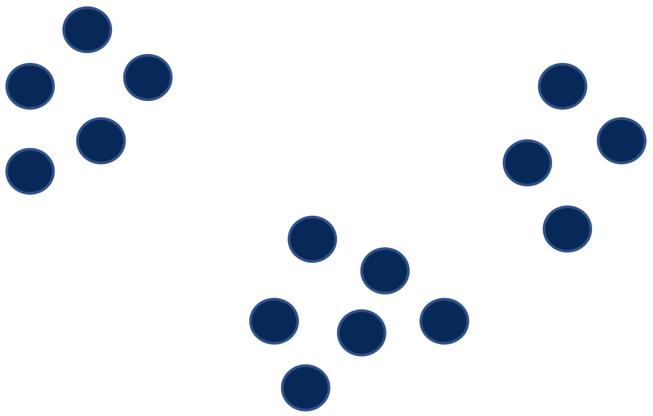
Evi Micha

Clustering



Clustering in ML/Data Analysis

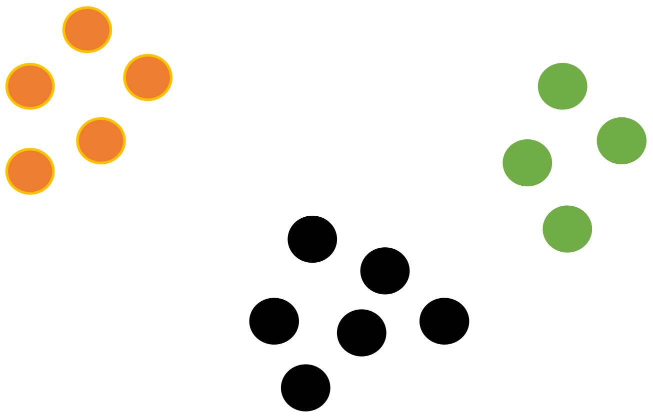
- Goal:
  - Analyze data sets to summarize their characteristics
  - Objects in the same group are similar



Clustering in ML/Data Analysis

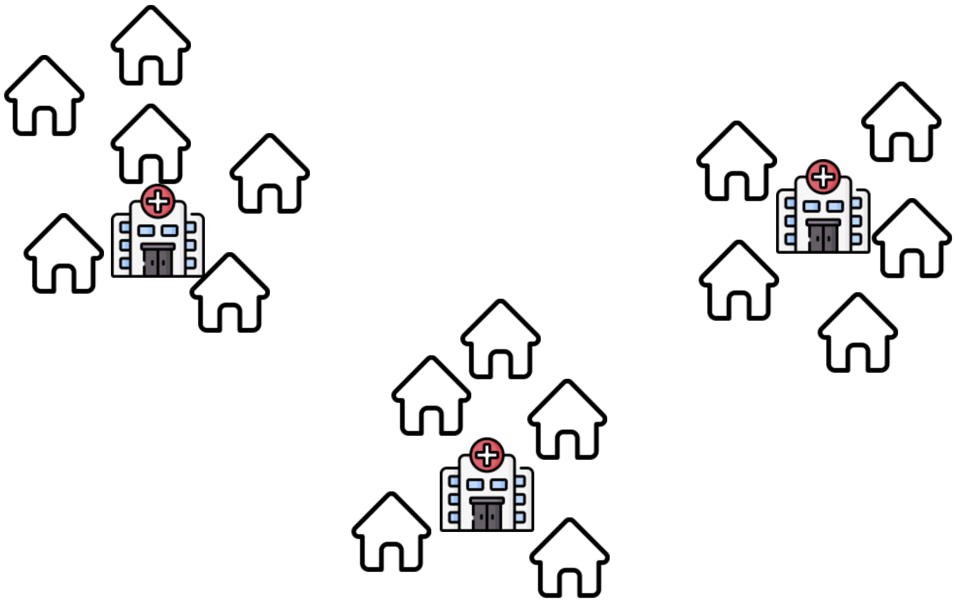
- Goal:
  - Analyze data sets to summarize their characteristics
  - Objects in the same group are similar

k=3



Clustering in Economics/OR

- Goal:
  - Allocate a set of facilities that serve a set of agents (e.g. hospitals)



Center-Based Clustering

- Input:
  - Set N of n data points
  - Set M of m feasible cluster centers
  - $\forall i, j \in N \cup M$: we have $d(i,j)$ (which forms a ***Metric Space***)
    - $\boldsymbol{d(i,i) = 0}$, $\forall\, i \in N \cup M$
    - $\boldsymbol{d(i,j) = d(j,i)}$, $\forall\, i, j \in N \cup M$
    - $\boldsymbol{d(i,j) \le d(i,\ell) + d(\ell,j)}$, $\forall\, i, j, \ell \in N \cup M$, (**Triangle Inequality**)
- Output:
  - A set $C \subseteq M$ of k centers, i.e. $C = \{c_1, \dots, c_k\}$
  - Each data point is assigned to its closest cluster center
    - $C(i) = argmin_{c \in C}\, d(i,c)$

Famous Objective Functions

- k-median: Minimizes the sum of the distances
  - $\min_{\substack{C \subseteq M: \\ |C| \leq k}} \sum_{i \in N} d(i, C(i))$
- k-means: Minimizes the sum of the square of the distances
  - $\min_{\substack{C \subseteq M: \\ |C| \leq k}} \sum_{i \in N} d^2(i, C(i))$
- k-center: Minimizes the maximum distance
  - $\min_{\substack{C \subseteq M: \\ |C| \leq k}} \max_{i \in N} d(i, C(i))$

Fairness in Clustering

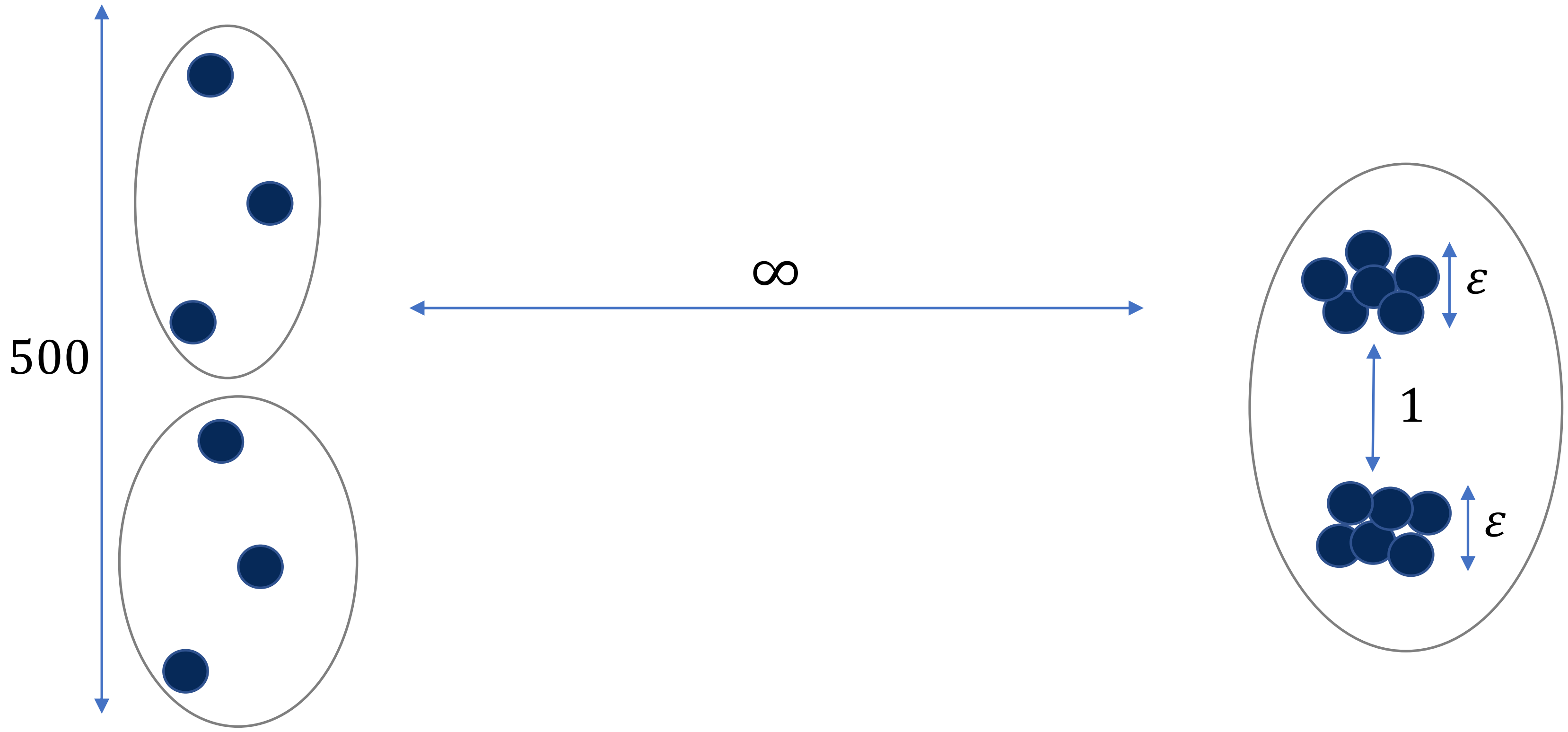
- Why do we need fairness:
  - Many decisions are made at least (partly) using algorithms

- Each point wishes to be as close as possible to some center

- **ML applications:** Closer to center ⇒ better represented by the center
- **FL applications:** Closer to the center ⇒ less travel distance to the facility

Fairness in Clustering

$k = 3$



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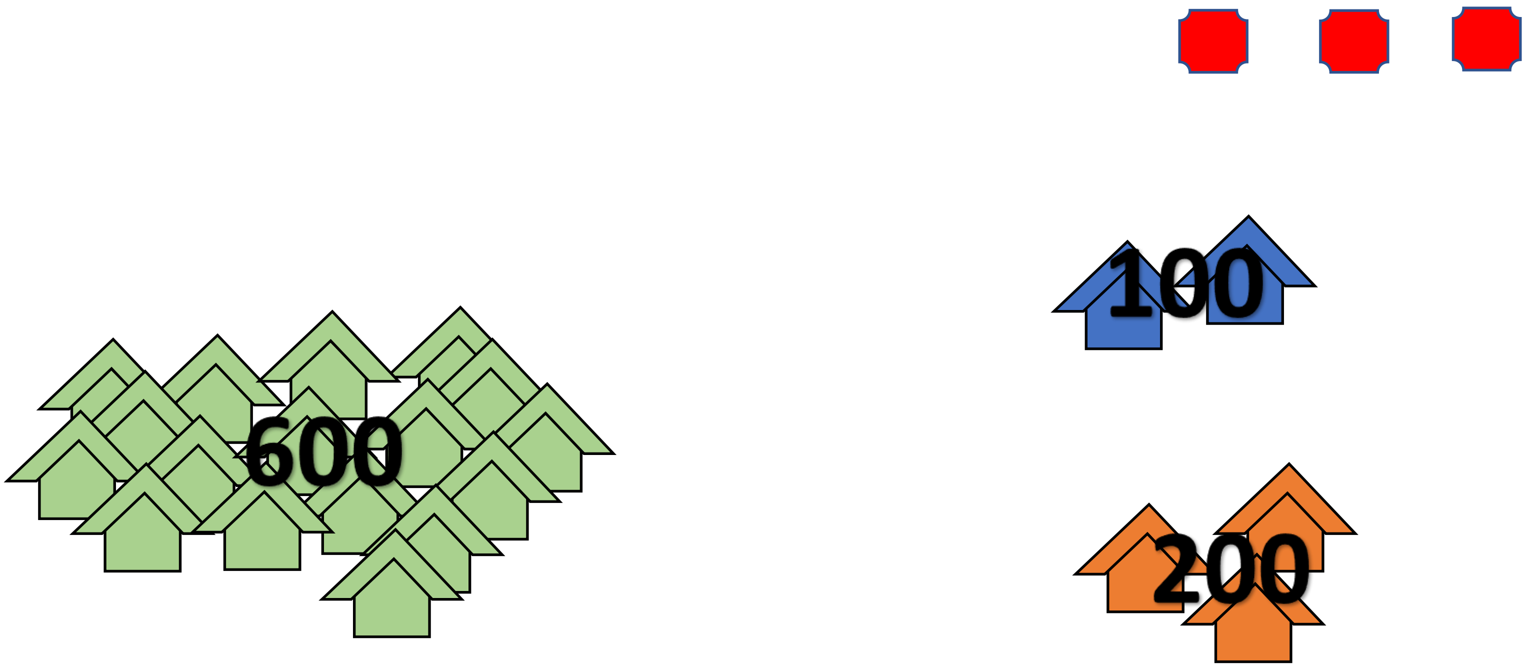

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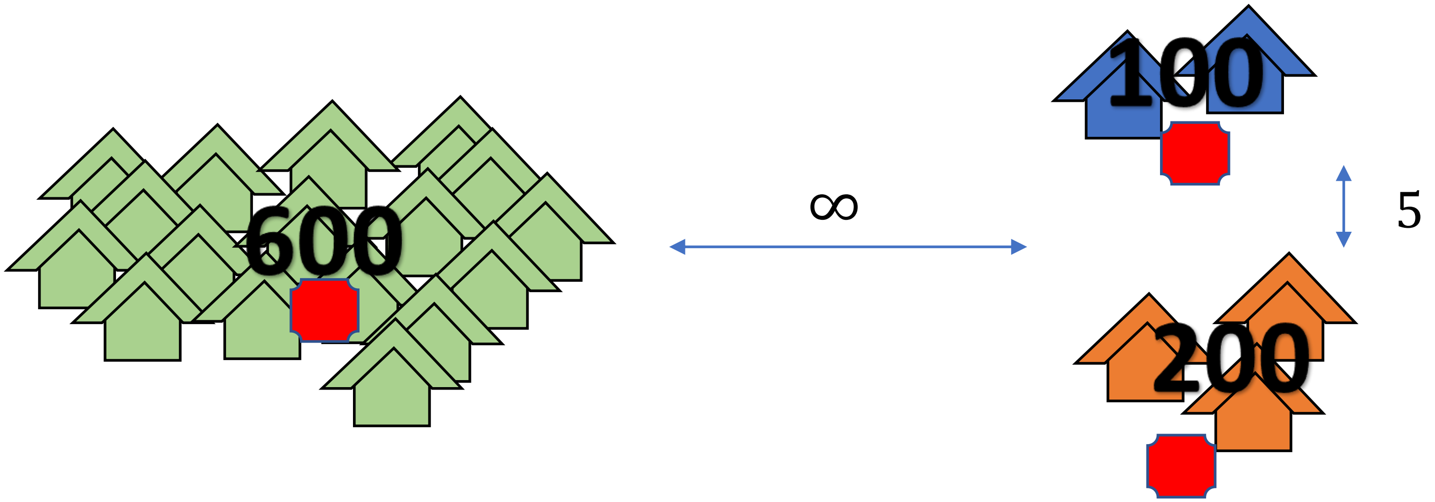

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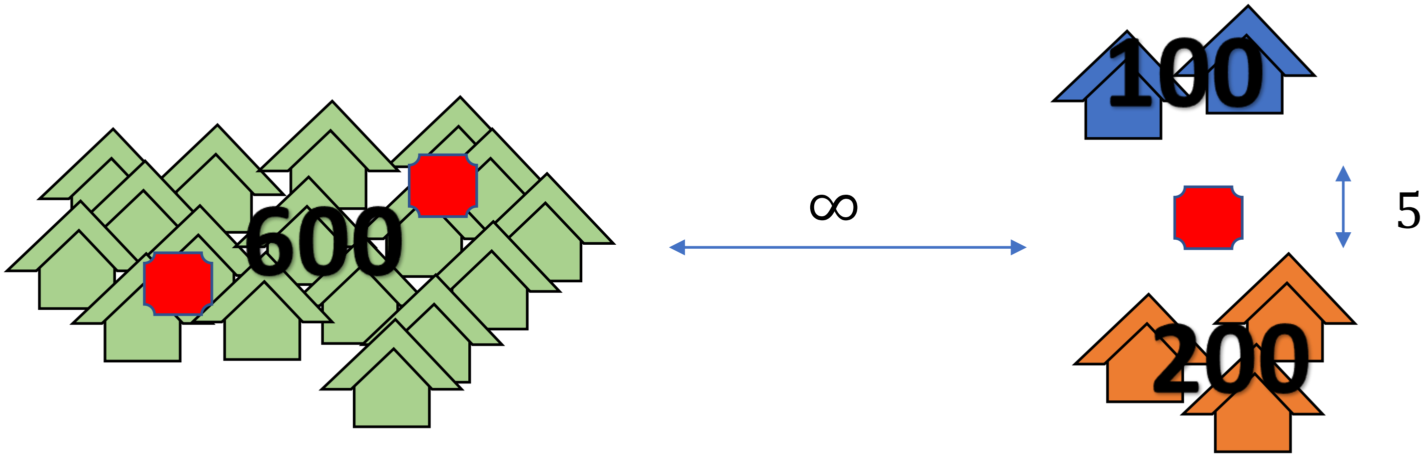

Fairness Through Proportionality

- *Proportionally Fair Clustering:*
  - *Every x% of the data points can select x% of the cluster centers*
  - *Every group of n/k agents "deserves" its own cluster center*

Core

- Definition in Committee Selection: W is in the core if
  - For all $S \subseteq N$ and $T \subseteq M$
  - If $|S| \geq |T| \cdot n/k$ (**large**)
  - Then, $|A_i \cap W| \geq |A_i \cap T|$ for some $i \in S$
  - "If a group can afford T, then T should not be a (strict) Pareto improvement for the group"

- Let $B(x, y)$ denotes the ball centered in x and has radius y
- Given clustering solution C, $C(i)$ denotes the closest center to $i \in N$

- Definition in Clustering: C is in the core if
  - For all $S \subseteq N$ and $y \subseteq M$
  - If $|S| \geq n/k$ (**large**)
  - Then, $d(i, C(i)) \leq d(i, y)$ for some $i \in S$
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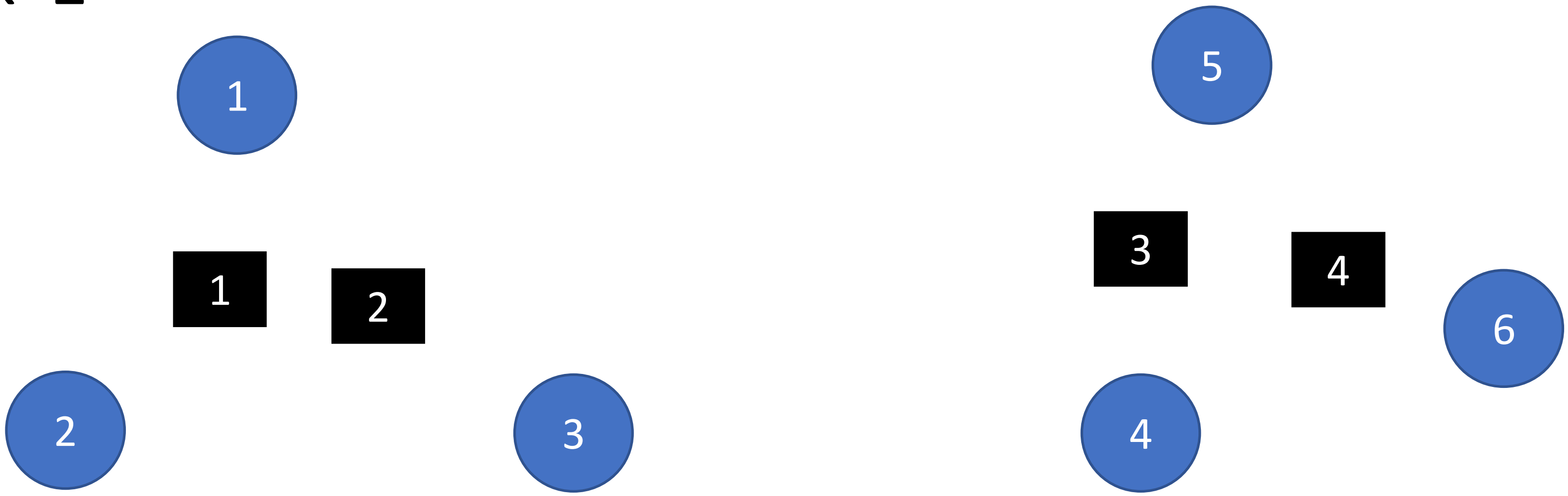
Core

Example

k=2



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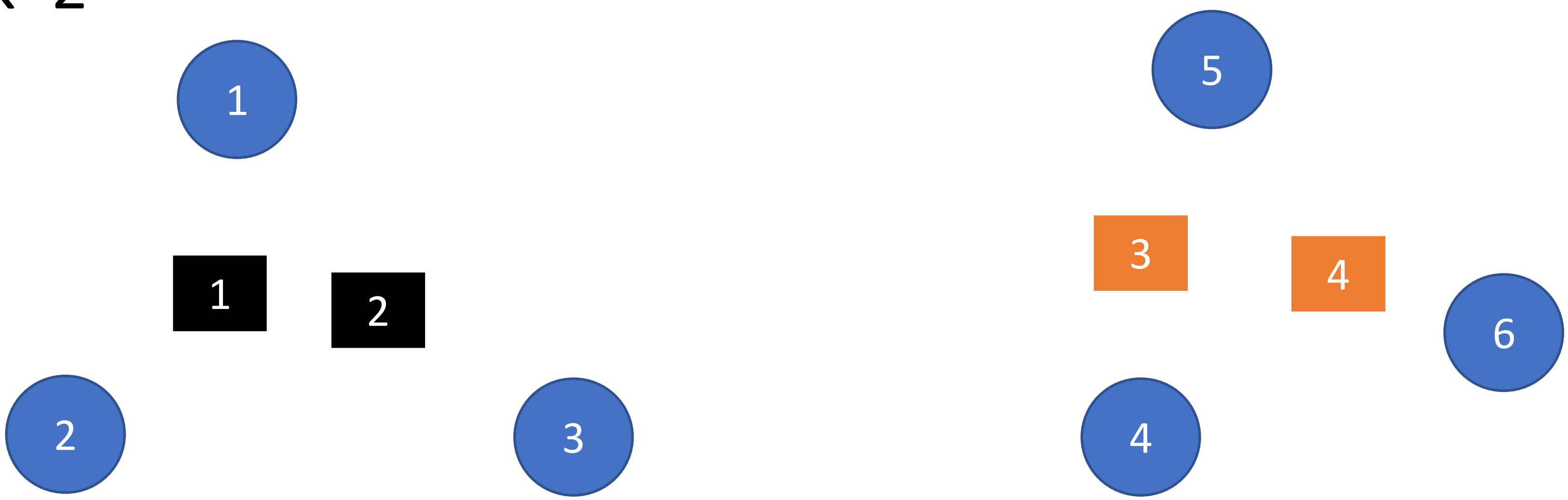

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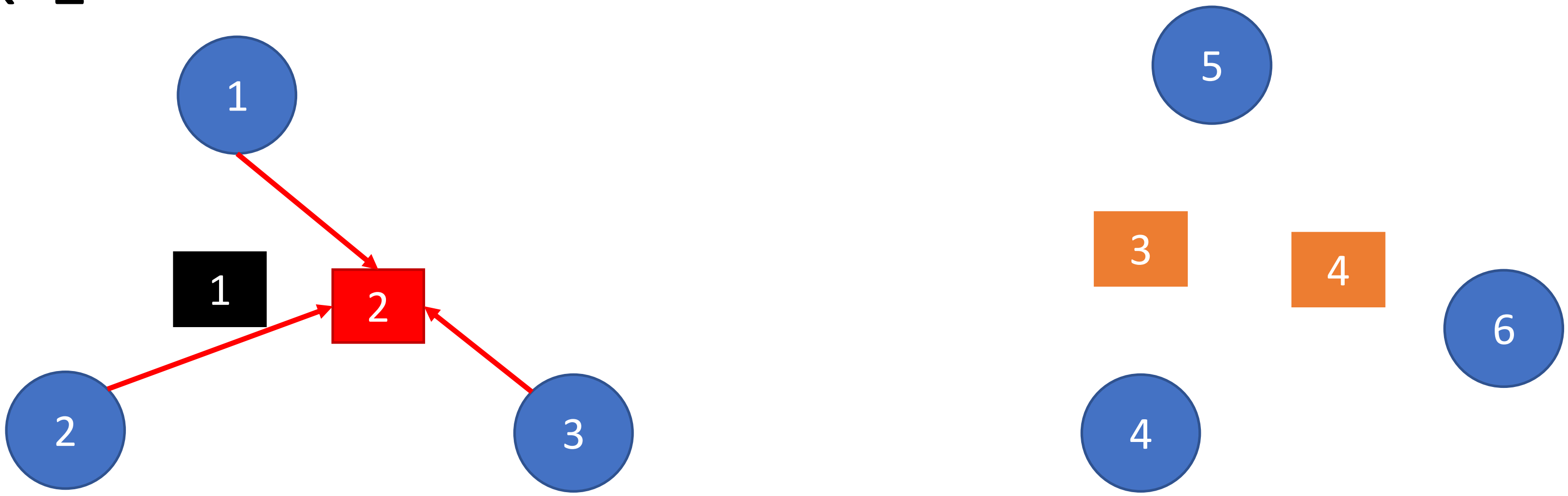

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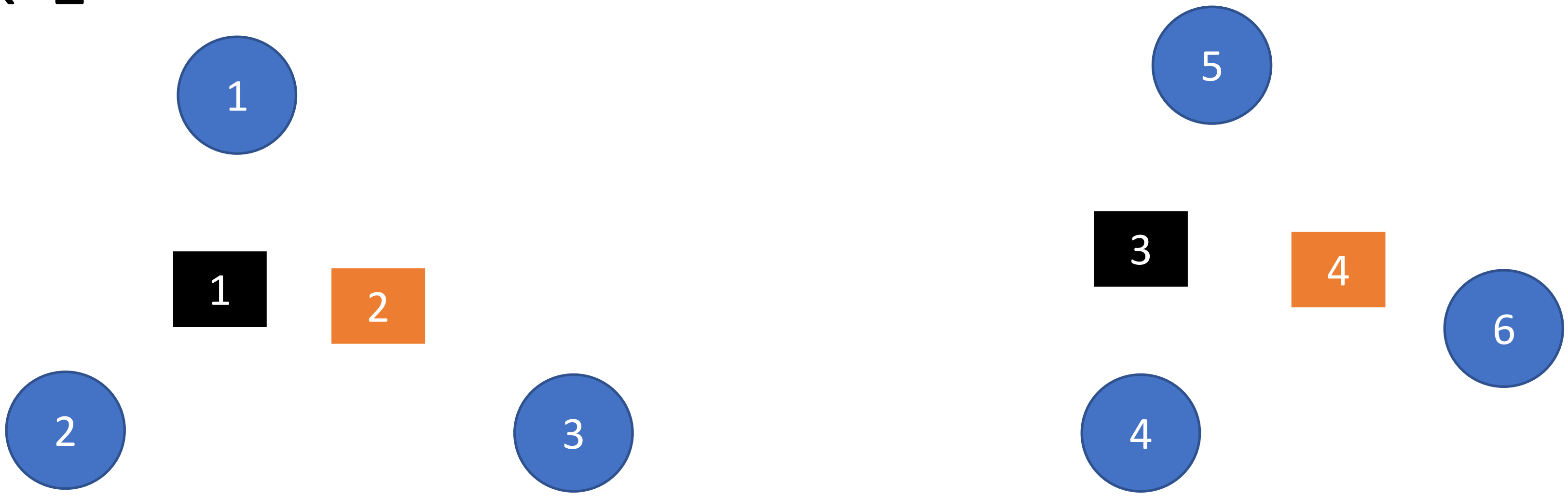

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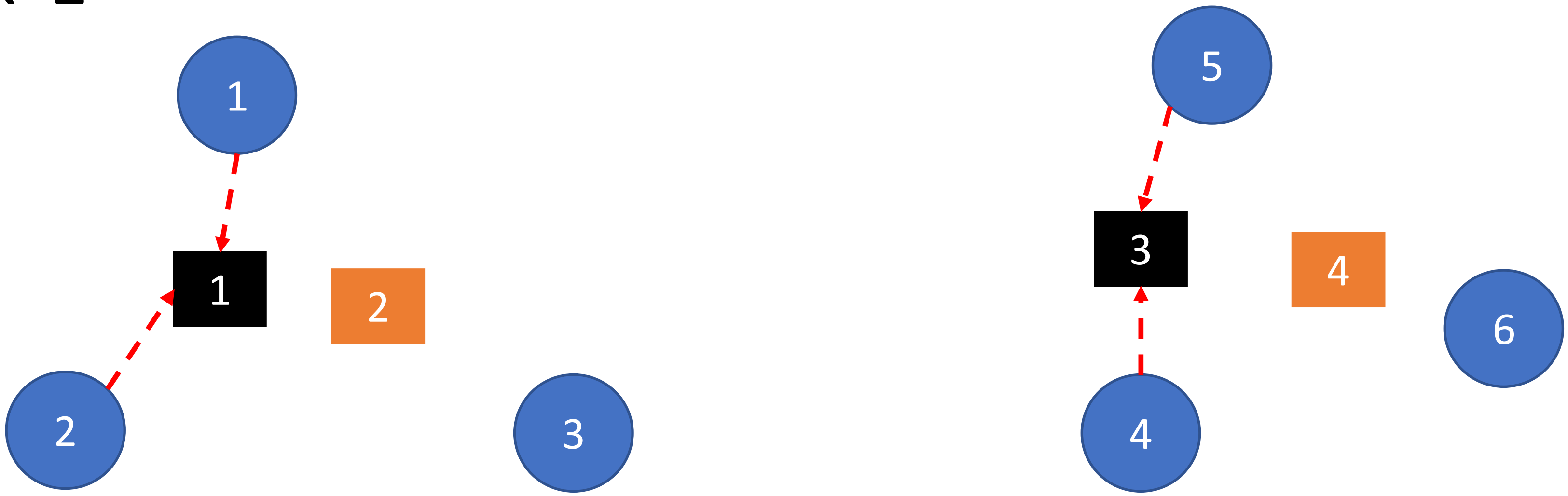

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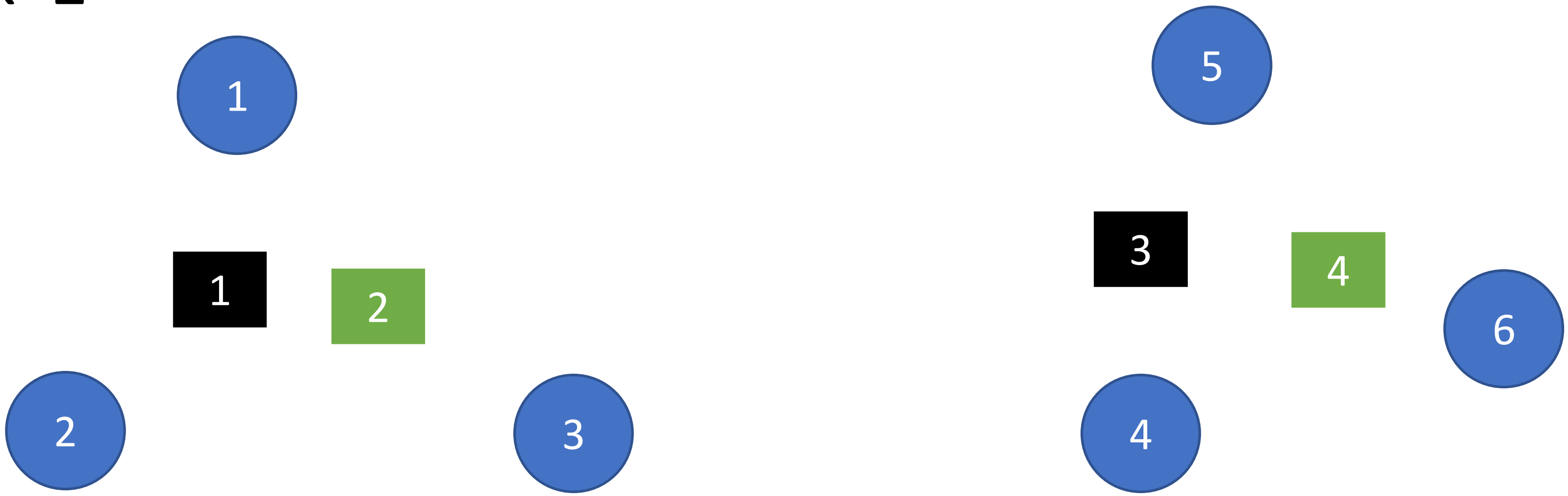
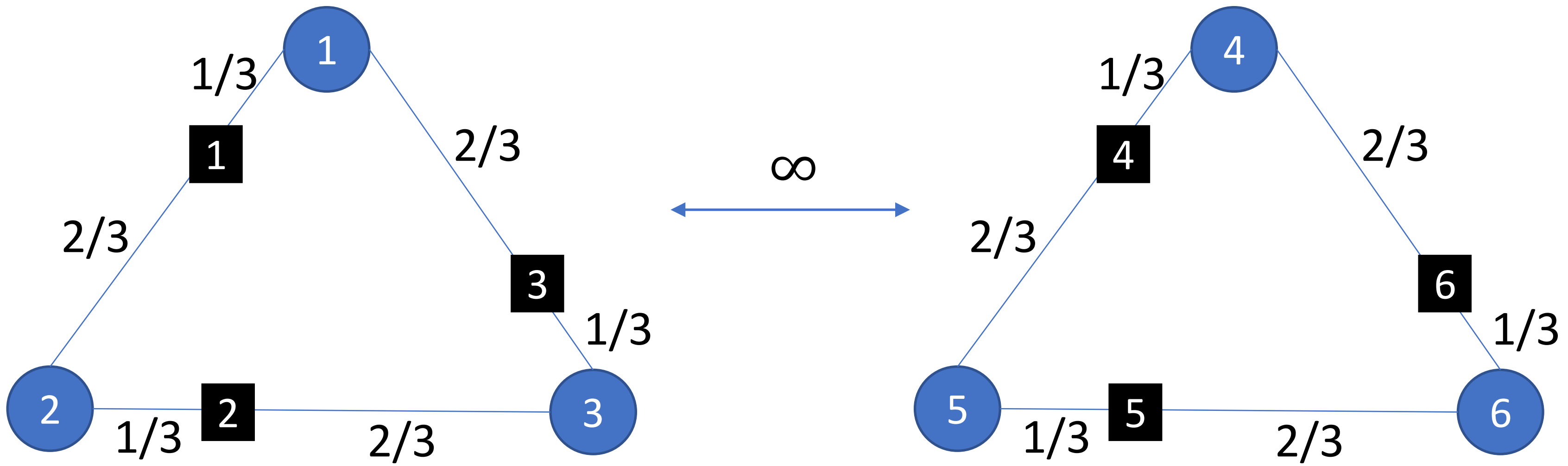


Core

- Theorem: A clustering solution in the core does not always exist

- Proof:

k=3



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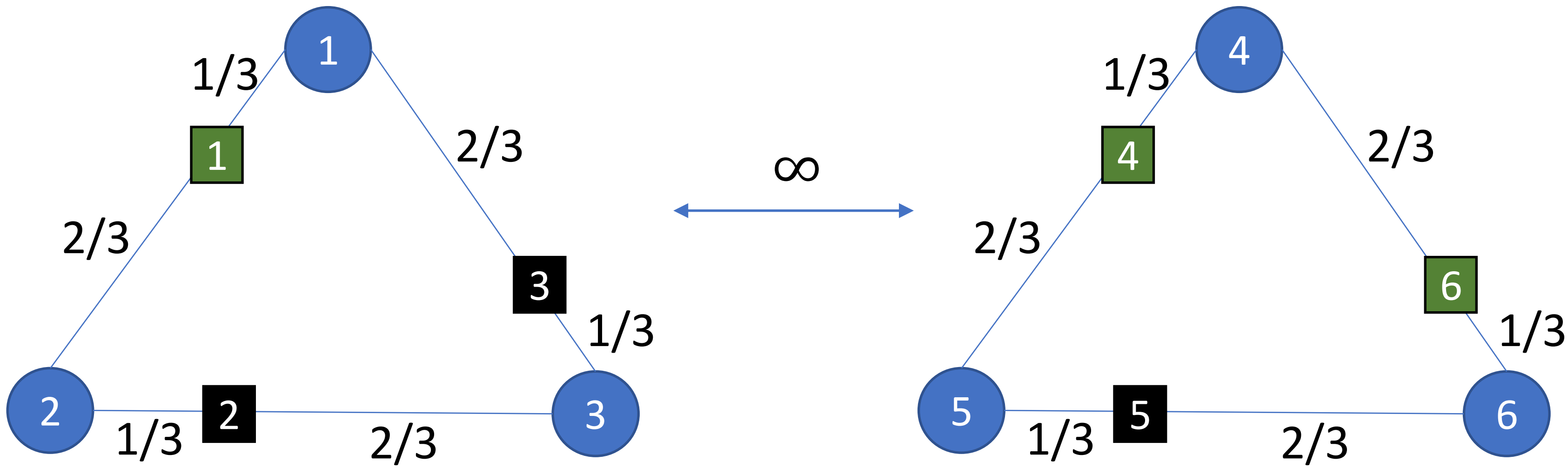

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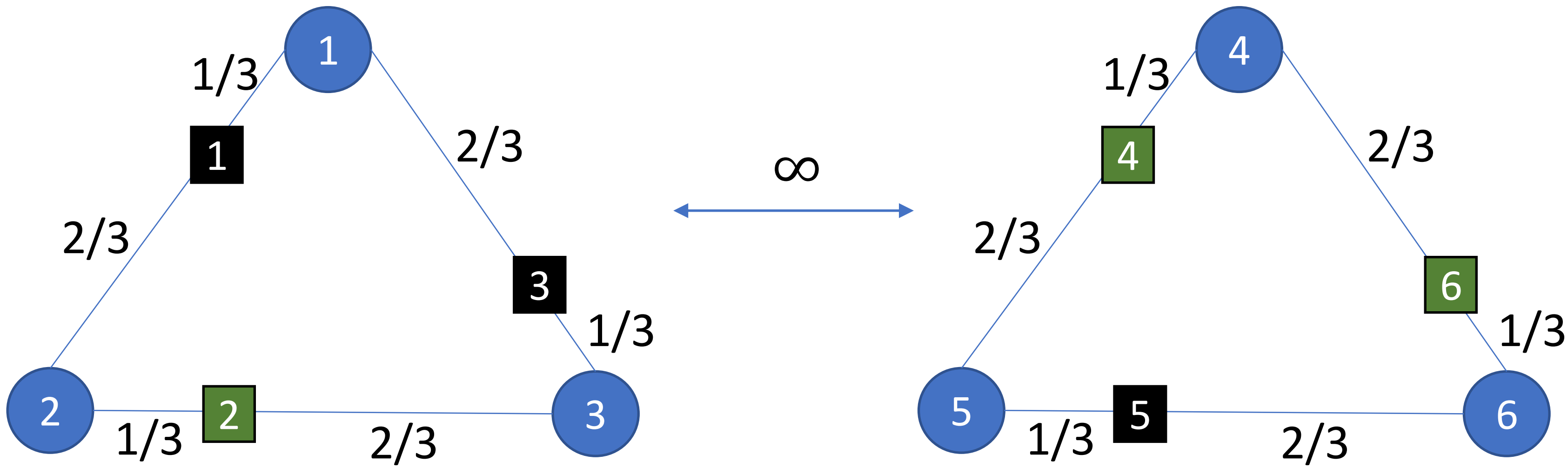

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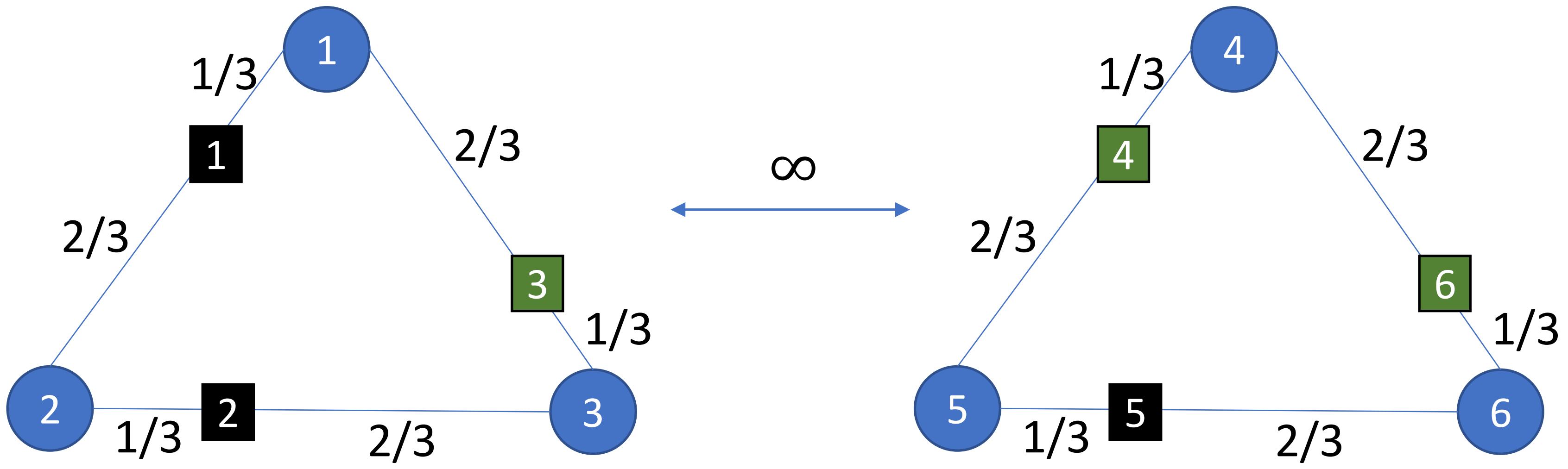


Core

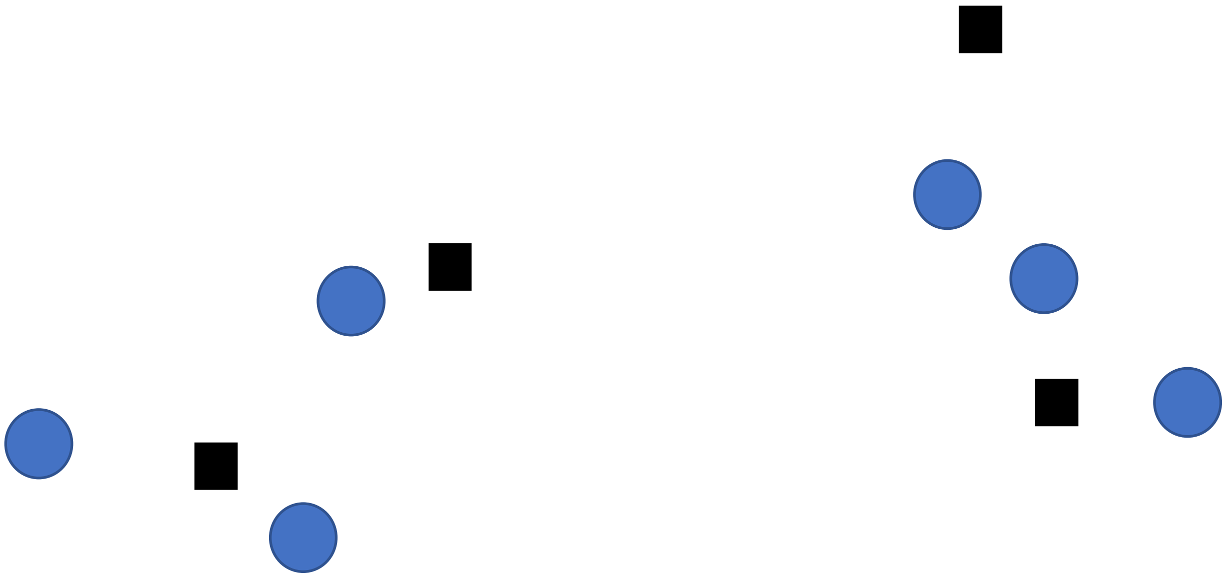
- Definition in Clustering: C is in the core if
  - For all $S \subseteq N$ and $y \subseteq M$
  - If $|S| \geq n/k$ (**large**)
  - Then, $d(i, C(i)) \leq \alpha \cdot d(i, y)$ for some $i \in S$
  - "If a group can afford a center y, then y should not be a (strict) Pareto improvement for the group"

α-Core:
A solution C is in the α-core, with $\alpha \geq 1$ if there is ***no*** *group of points $S \subseteq N$ with $|S| \geq n/k$ and $y \in M$ such that:*

$$\forall i \in S, \alpha \cdot d(i, y) < d(i, C(i))$$

Greedy Capture

$k = 3$



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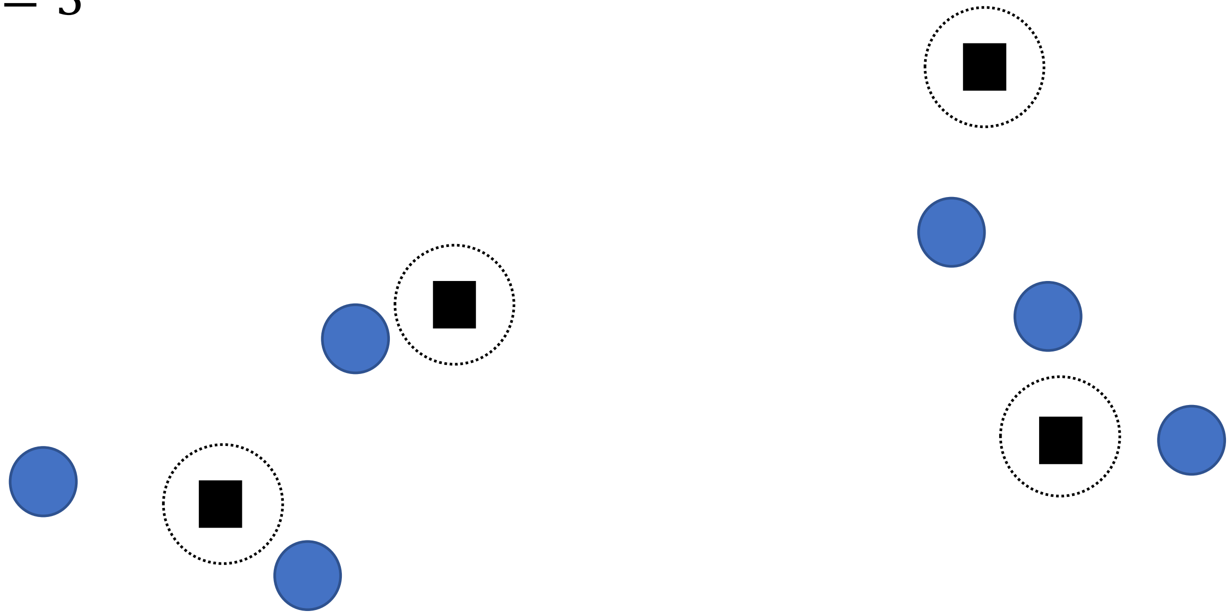

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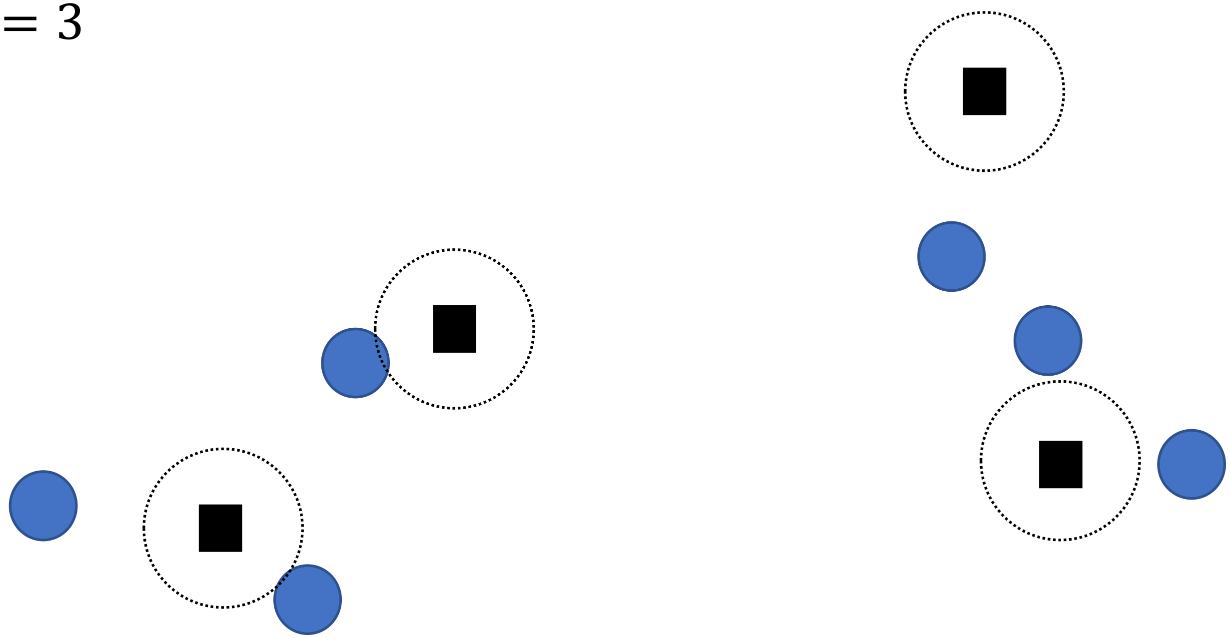

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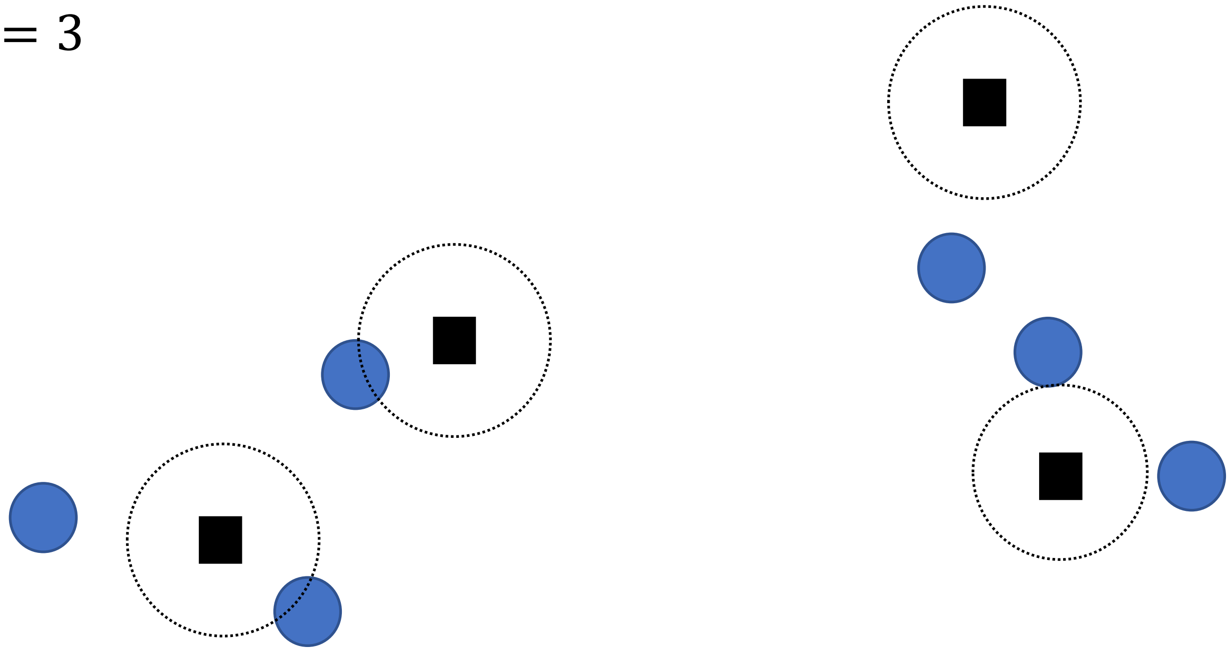

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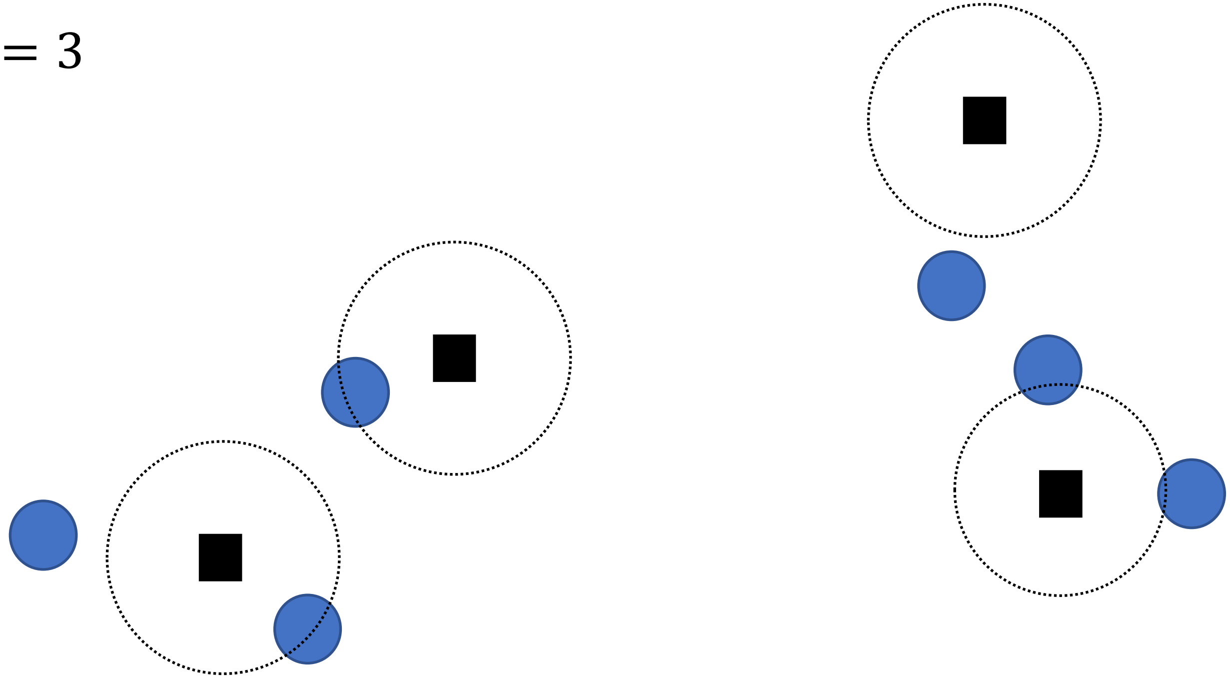

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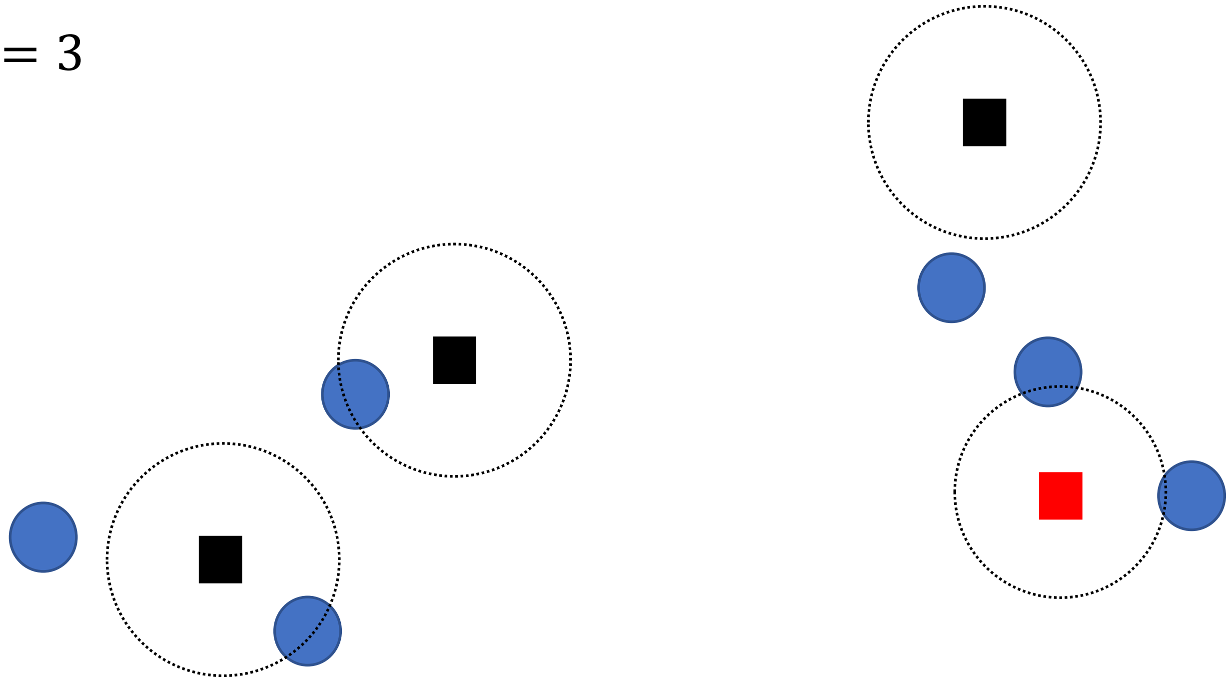

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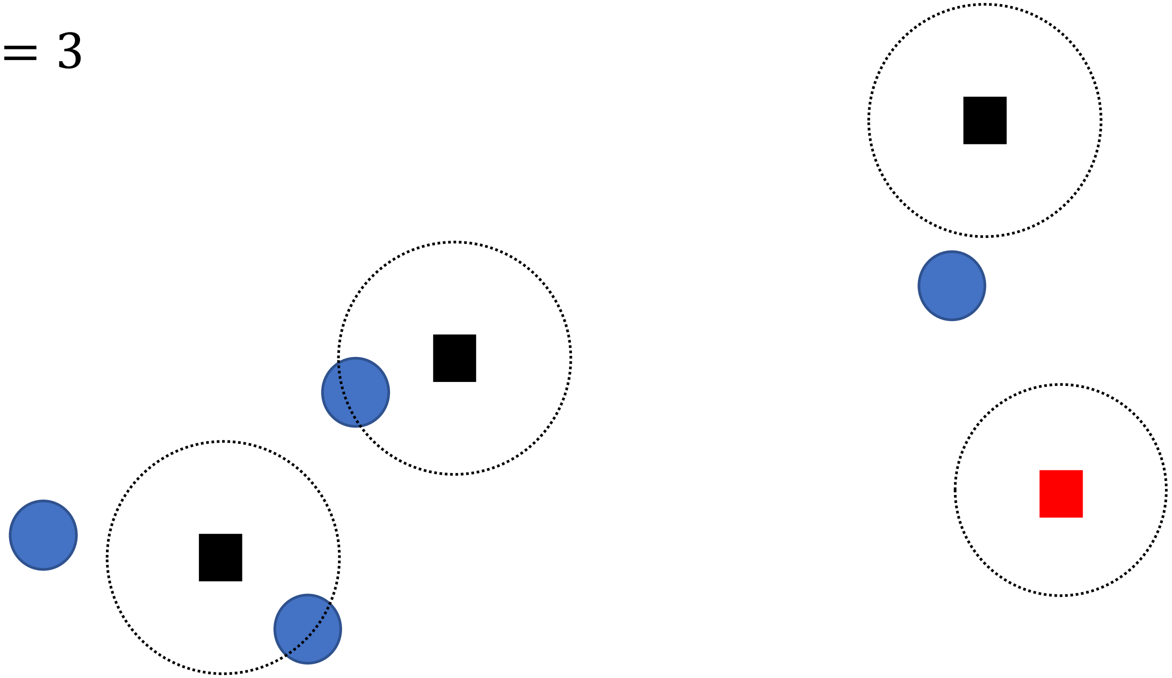

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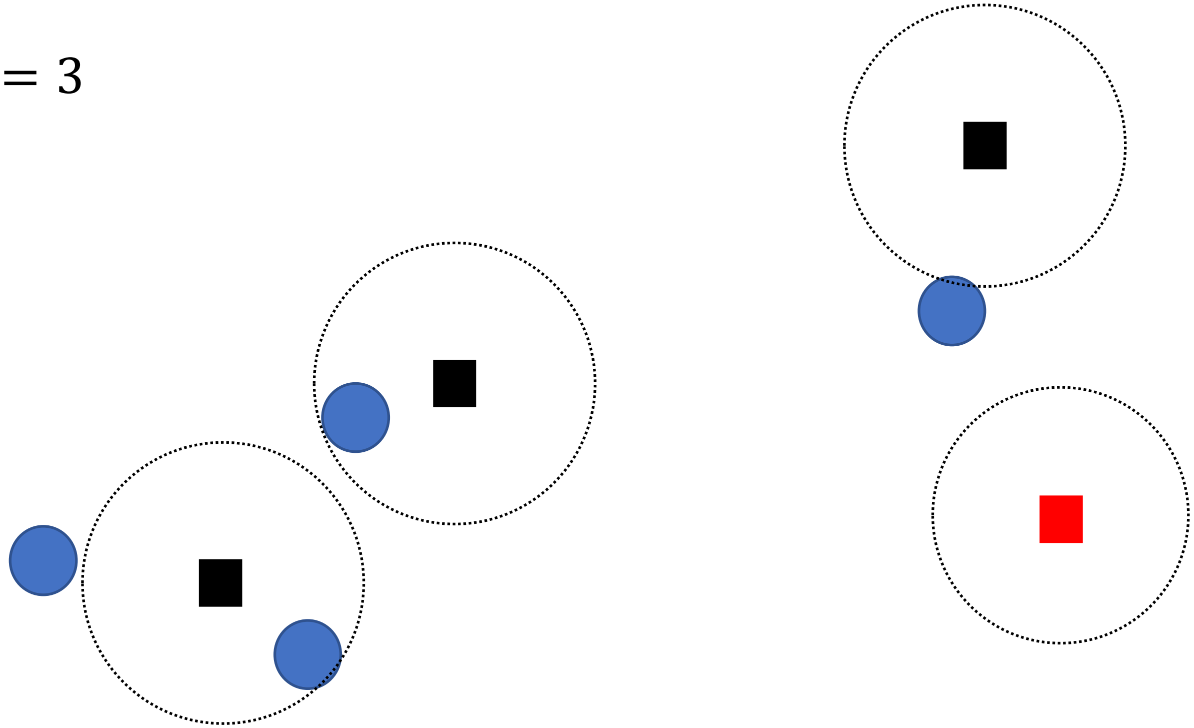

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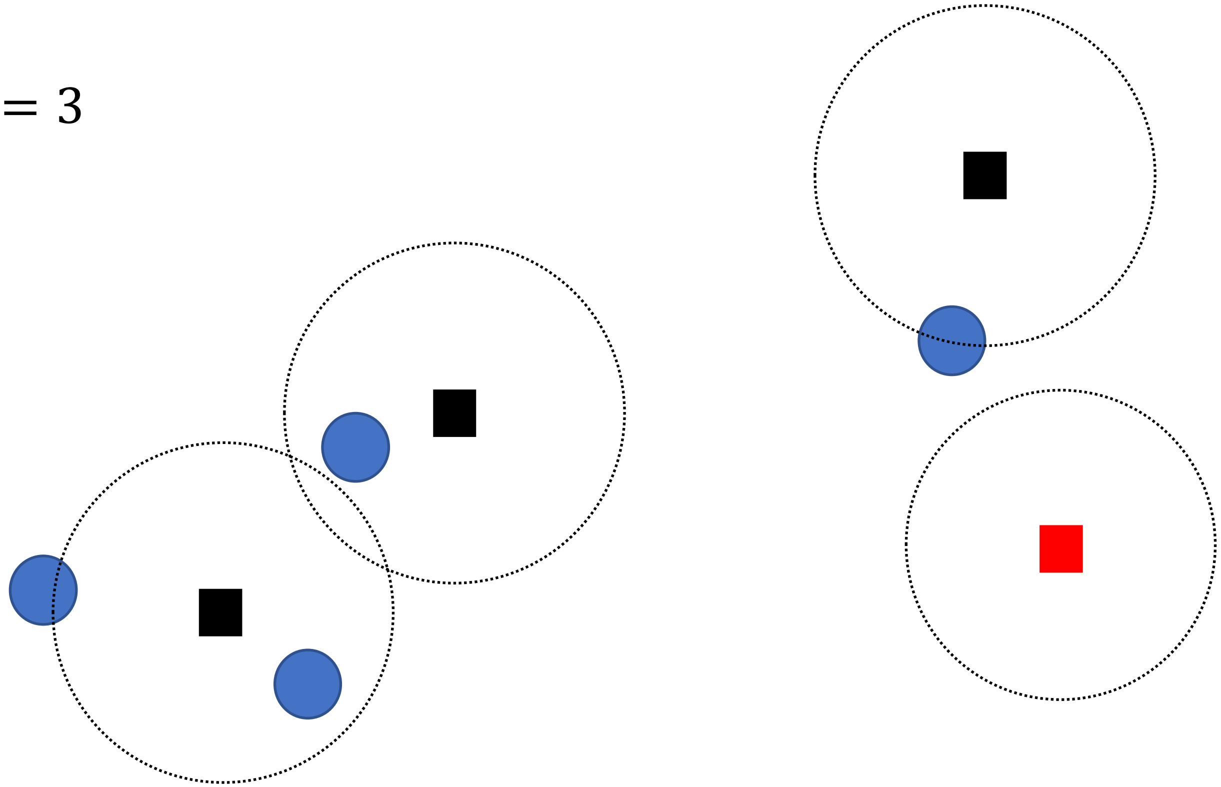

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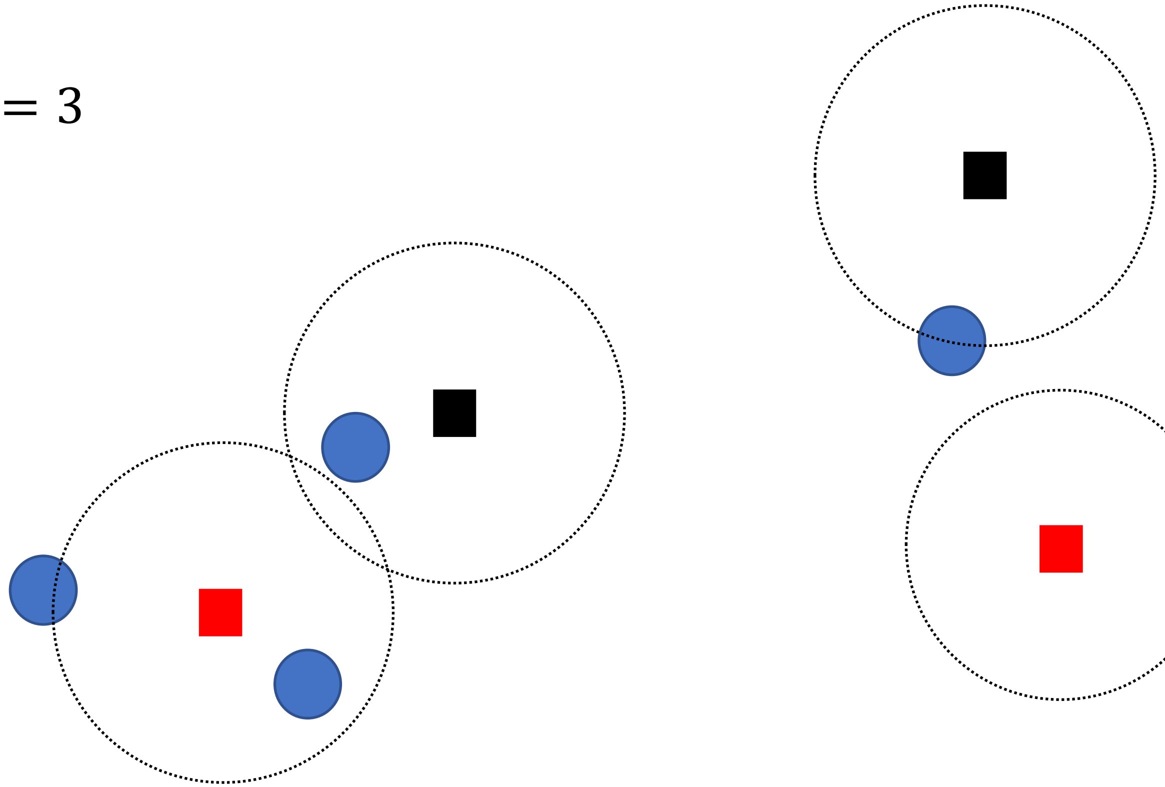

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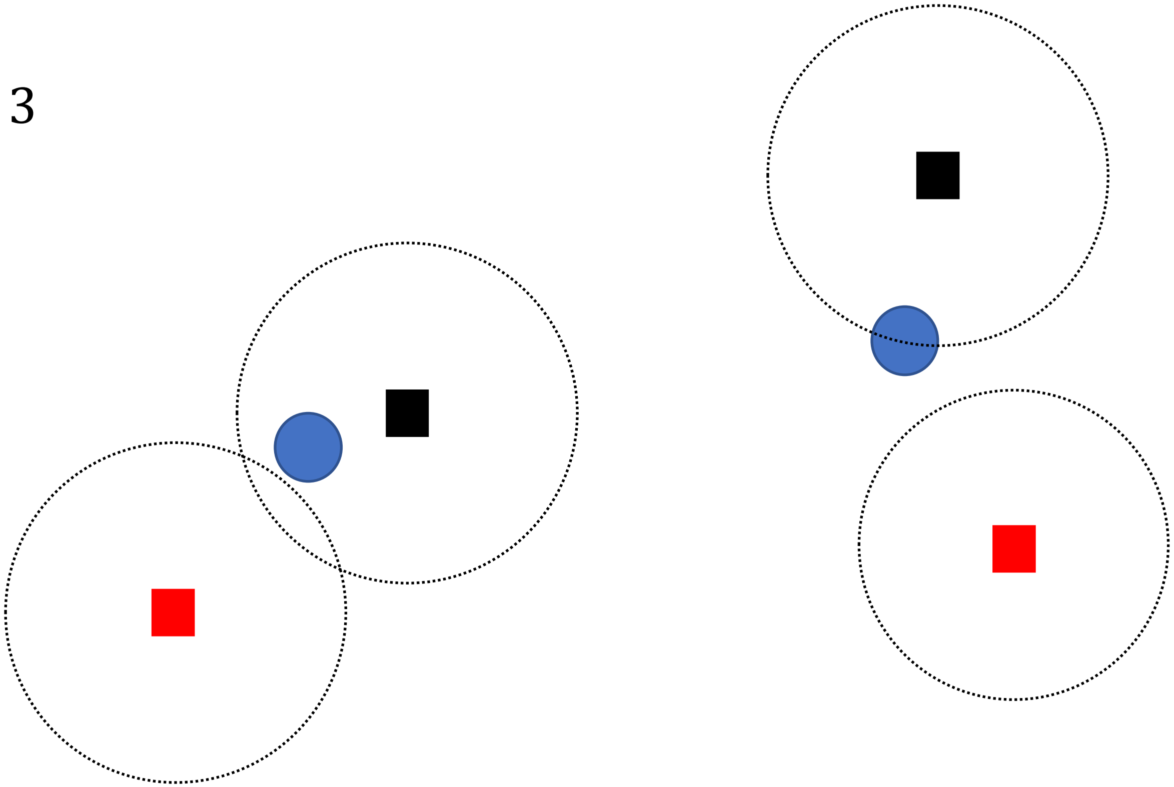

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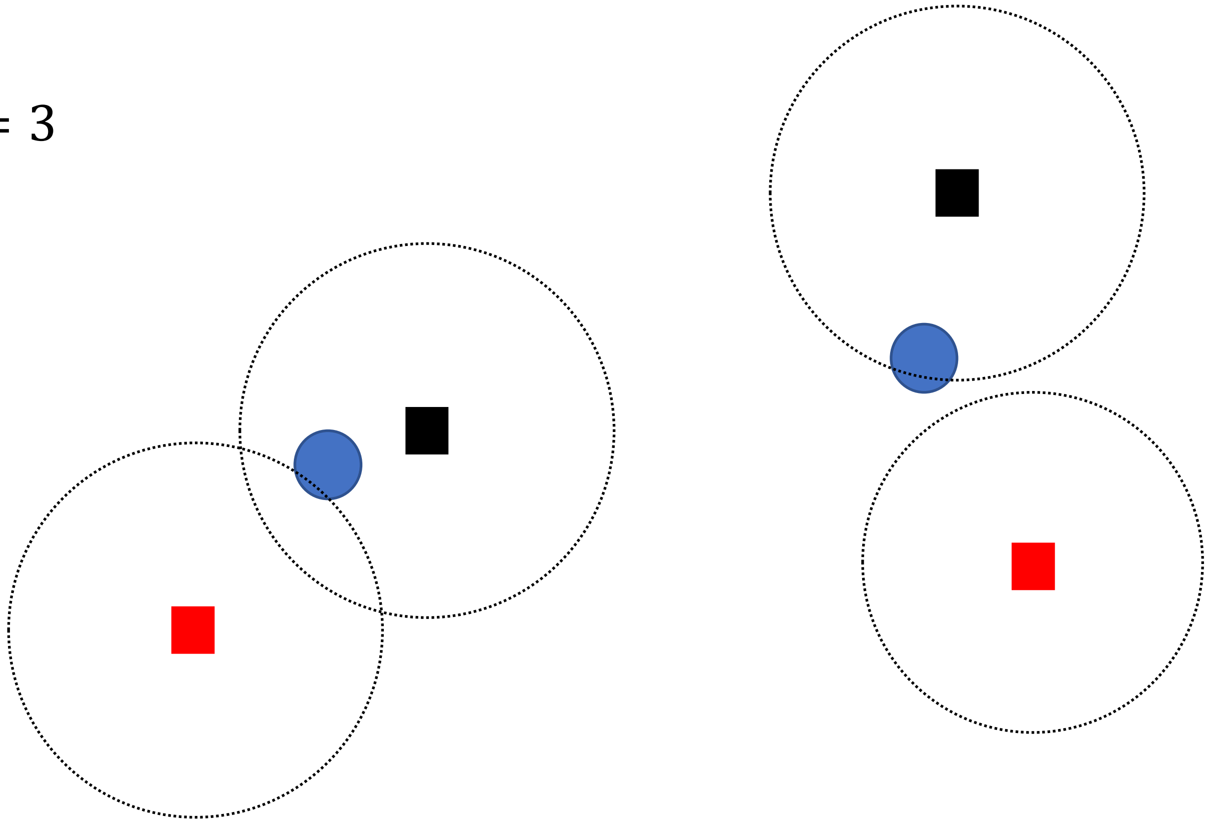

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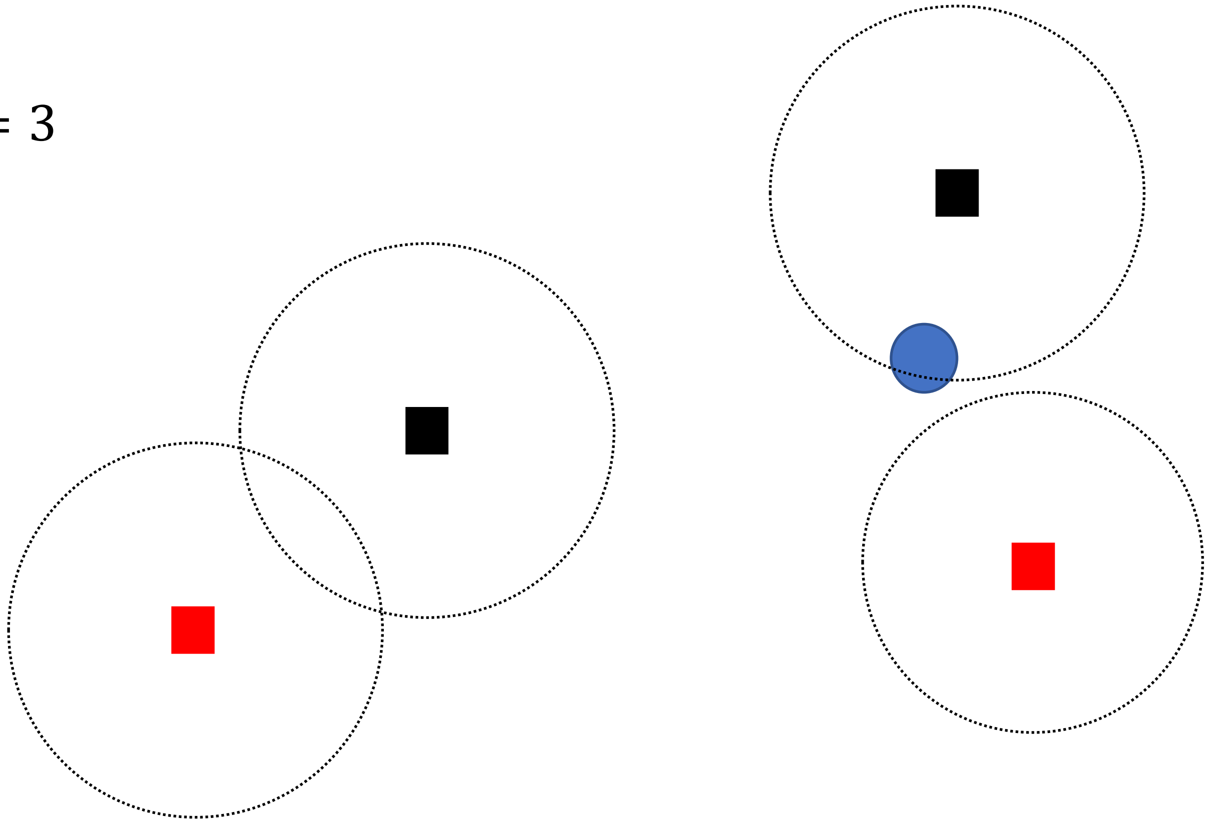

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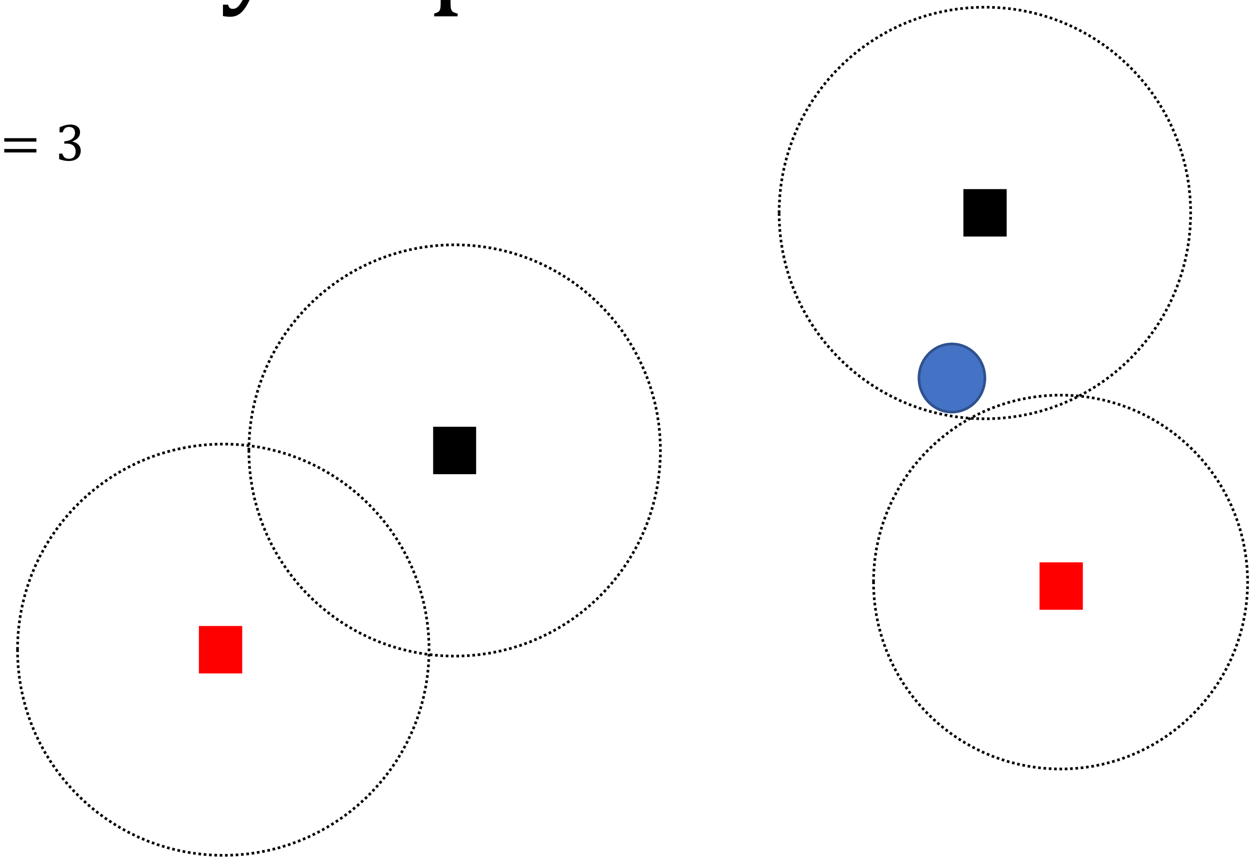

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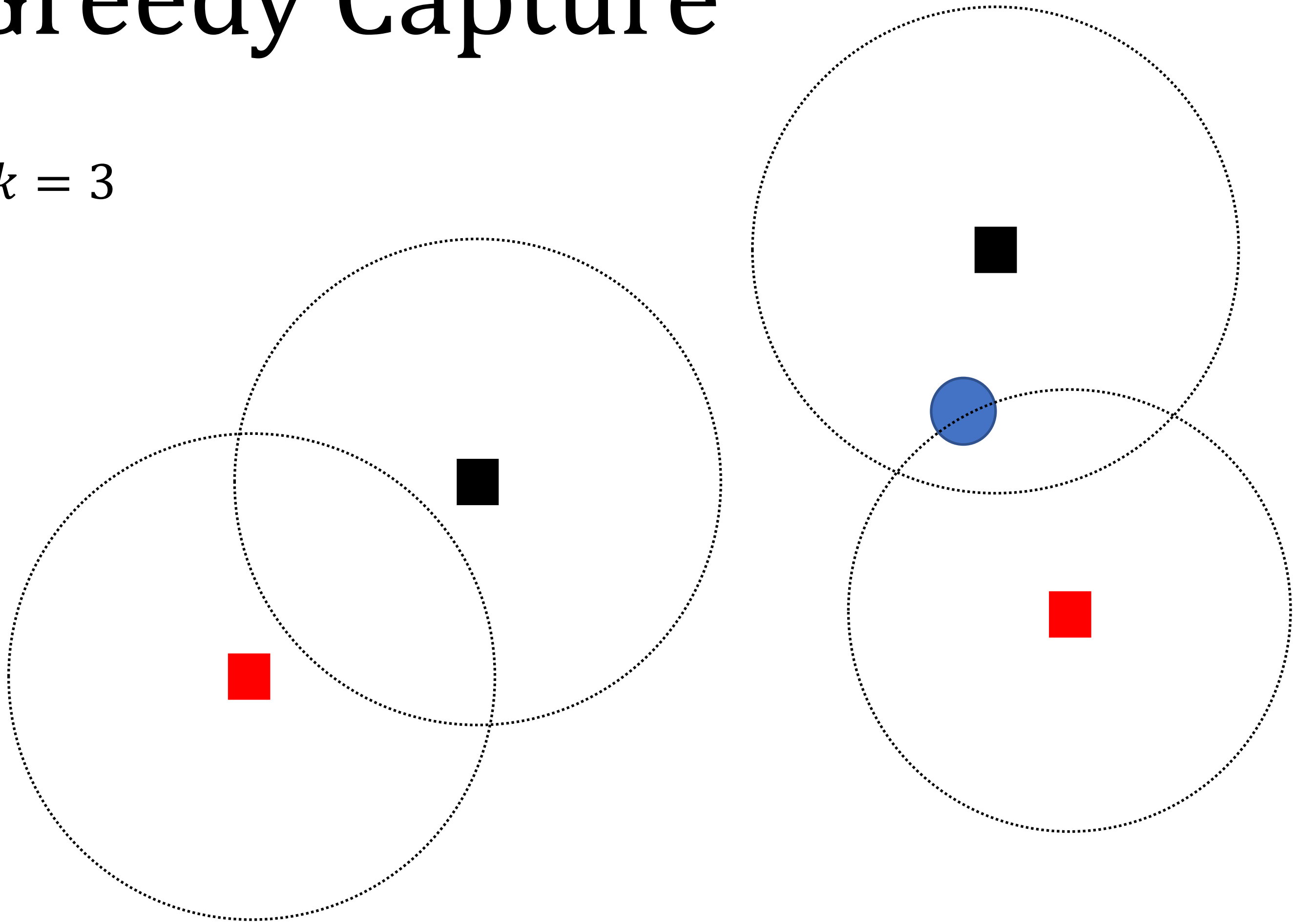

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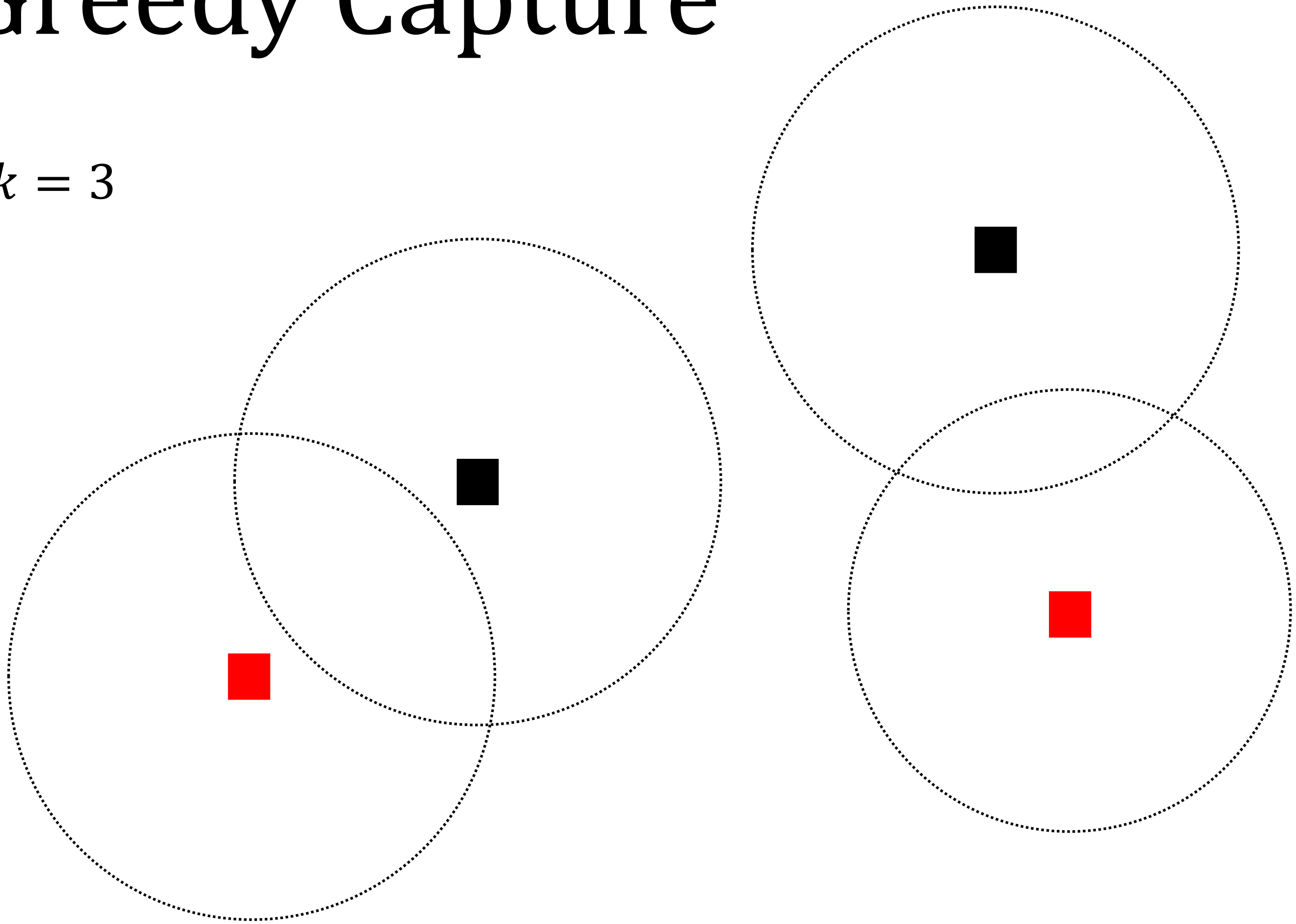

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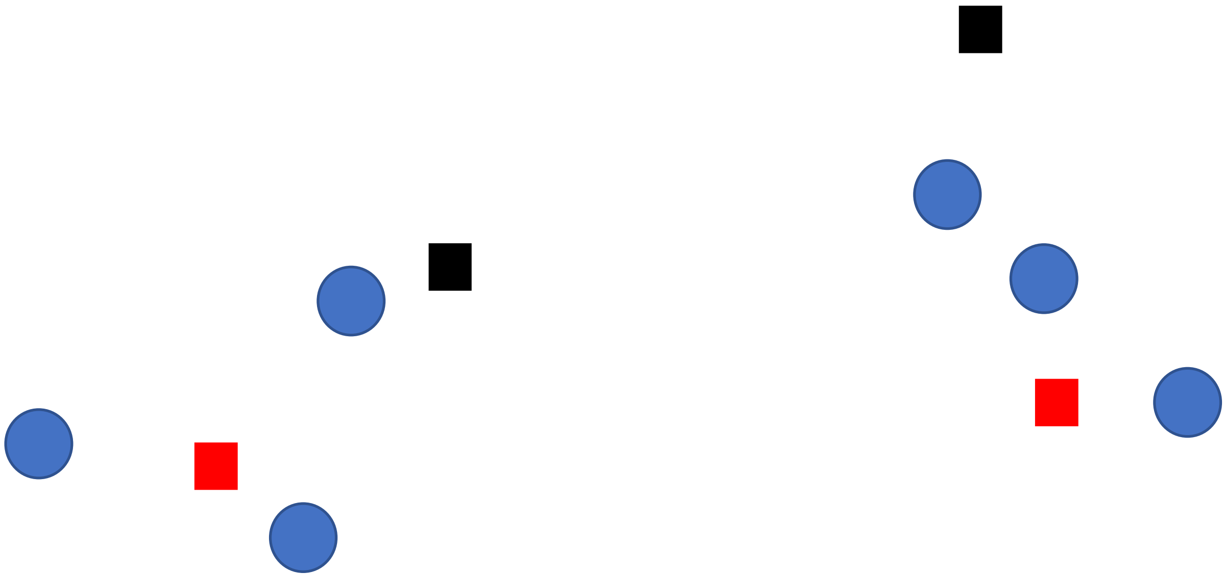


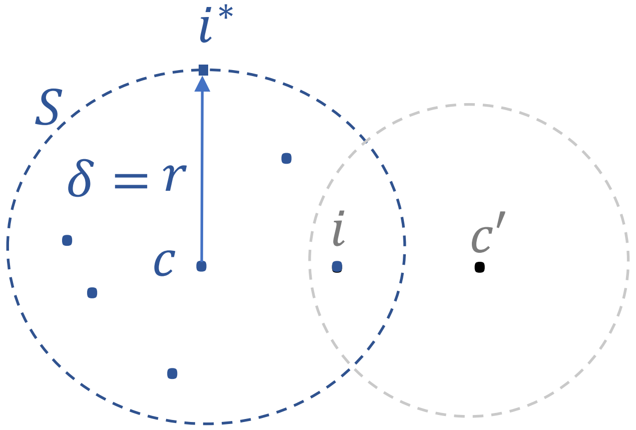
Greedy Capture

Greedy Capture

1. $\delta \leftarrow 0$; $C \leftarrow 0$
2. *While* $N \neq 0$ *do*
3. *Smoothly increase* δ
4. *While* $\exists\, c \in C$ *such that* $|B(c,\delta) \cap N| \geq 1$ *do*
5. $C: N \leftarrow N \setminus (B(c,\delta) \cap N)$
6. *While* $\exists\, c \in M \setminus C$ *such that* $|B(c,\delta) \cap N| \geq n/k$ *do*
7. $C \leftarrow C \cup c$
8. $N \leftarrow N \setminus (B(c,\delta) \cap N)$
9. *Return* C

Greedy Capture

- Theorem [Chen et al. '19]: Greedy Capture returns a clustering solution in the $(1+\sqrt{2})$-core.
- Proof:
- Let C be the solution that Greedy Capture returns
- Suppose for contradiction that there exists $S \subseteq N$, with $|S| \geq \frac{n}{k}$ and $c \in M \setminus C$, such that $\forall i \in S, (1+\sqrt{2}) \cdot d(i,c) < d(i, C(i))$



$$\min\left(\frac{d(i,c')}{d(i,c)}, \frac{d(i^*,c')}{d(i^*,c)}\right)$$

$$\leq \min\left(\frac{d(i,c')}{d(i,c)}, \frac{d(i^*,c)+d(c,c')}{d(i^*,c)}\right) \text{ (triangle inequality)}$$

$$\leq \min\left(\frac{d(i,c')}{d(i,c)}, \frac{d(i^*,c)+d(c,i)+d(i,c')}{d(i^*,c)}\right) \text{ (triangle inequality)}$$

$$\leq \min\left(\frac{d(i^*,c)}{d(i,c)}, 2+\frac{d(i,c)}{d(i^*,c)}\right) \ (d(i,c') \leq d(i^*,c))$$

$$\leq \max_{z \geq 0}(\min(z, 2+1/z)) \leq 1+\sqrt{2}$$

$\blacksquare$

Justified Representation

- Definition in Committee Selection: W satisfies JR if
  - For all $S \subseteq N$
  - If $|S| \geq n/k$ (large) and $|\cap_{i \in S} A_i| \geq 1$ (cohesive)
  - Then, $|A_i \cap W| \geq 1$ for some $i \in S$
  - "If a group deserves one candidate and has a commonly approved candidate, then not every member should get 0 utility"

- Definition in Clustering: C satisfies JR if
  - For all $S \subseteq N$
  - If $|S| \geq n/k$ (large) and $| \cap_{i \in S} B(i,r) \cap M| \geq 1$ (cohesive)
    - i.e. $\forall i \in S, d(i,c) \leq r$ for some $c \in M$
  - Then, $|B(i,r) \cap C| \geq 1$ for some $i \in S$
    - i.e. $d(i, C(i)) \leq r$ for some $i \in S$
  - "If a group deserves one cluster center and has a center that has distance at most r from each of them, then not every member should have distance larger than r from all the centers in the clustering "

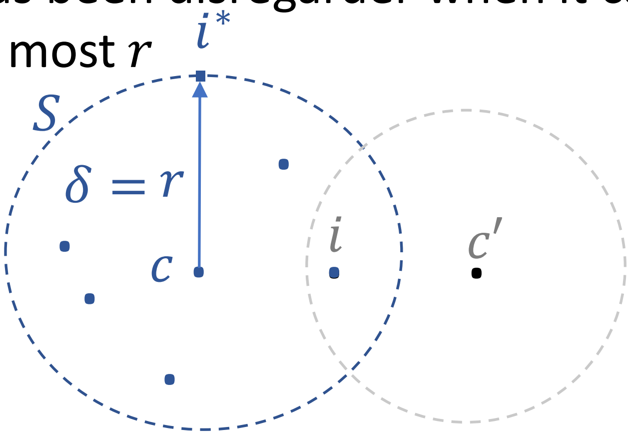
Justified Representation

- Question: What is the relationship between JR and core in clustering?

1. core $\Rightarrow$ JR
2. JR $\Rightarrow$ core
3. JR$=$core
4. JR $\neq$ core

Justified Representation

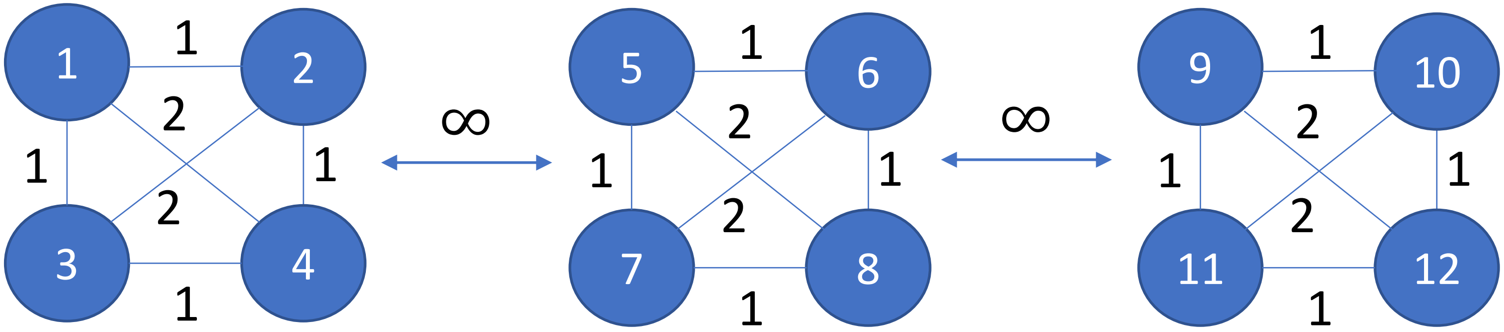
- Theorem [Kellerhals and Peters '24]: Greedy Capture returns a clustering solution that is JR
- Proof:
- Let C be the solution that Greedy Capture returns
- Suppose for contradiction that there exists $S \subseteq N$, with $|S| \geq \frac{n}{k}$ and $c \in M \setminus C$, such that $\forall i \in S,\ d(i,c) \leq r$ and $d\big(i, C(i)\big) > r$
- If none of $i \in S$ has been disregarded, then $|B(c,\delta)| \geq n/k$ and then c is included in the committee
- Otherwise, some of $i \in S$ has been disregarder when it captured from a ball centered at c with radius at most r



Individual Fairness

- Definition: C satisfies Individual Fairness (IF) if
  - $N = M$
  - Let $r_i = \min_{r \in \mathbb{R}} \{ |B(i,r) \cap N| \geq n/k \}$
  - For all $i \in N$, $|B(i, r_i) \cap C| \geq 1$
  - "Each individual expects a center within their proportional neighborhood"

- Theorem [Jung et al. '19]: An individually fair clustering solution does not always exist
- Proof: $k = 4$



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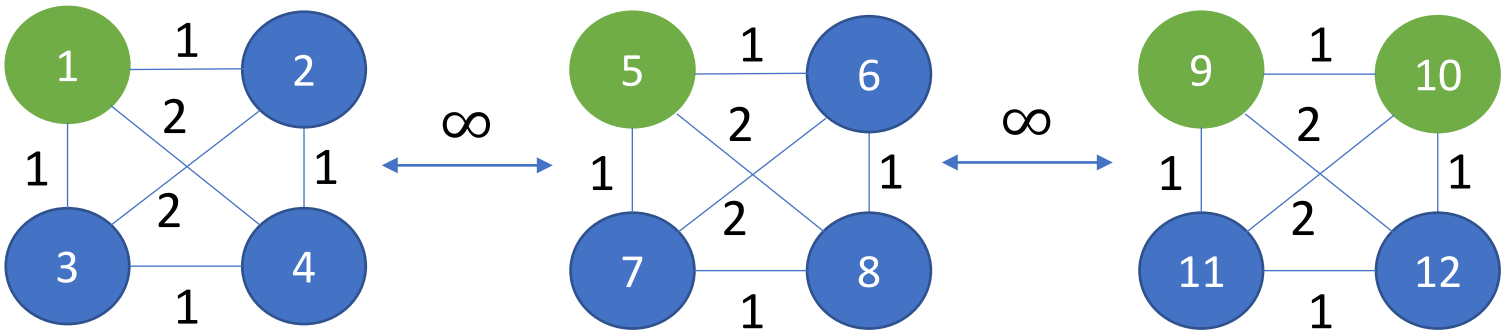

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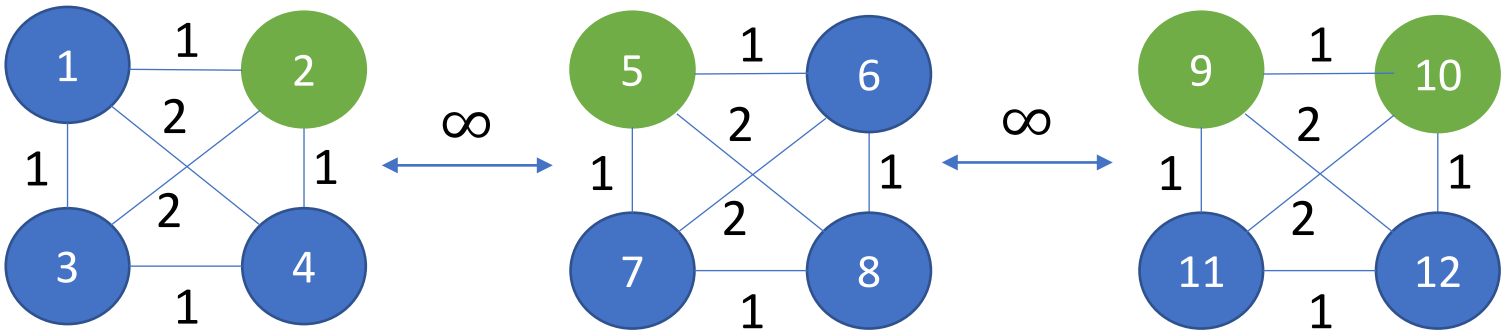

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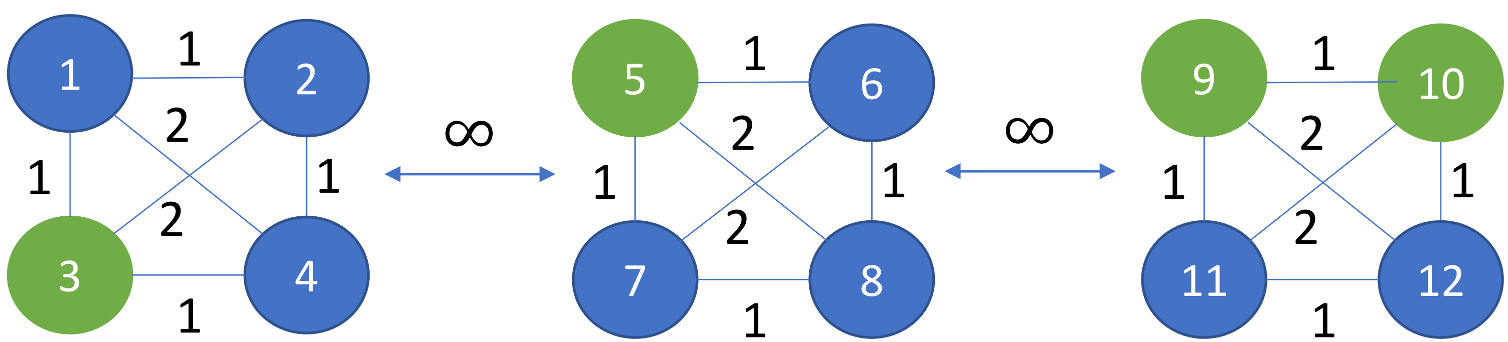

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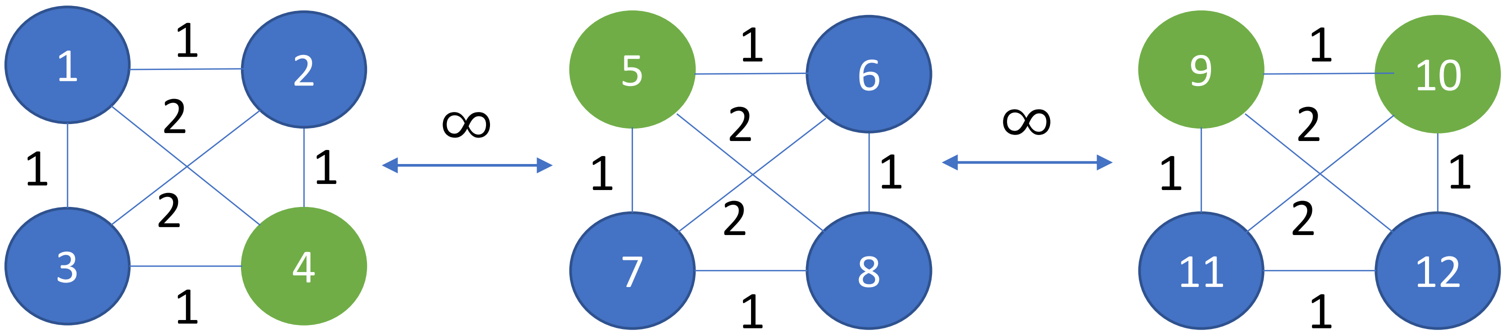
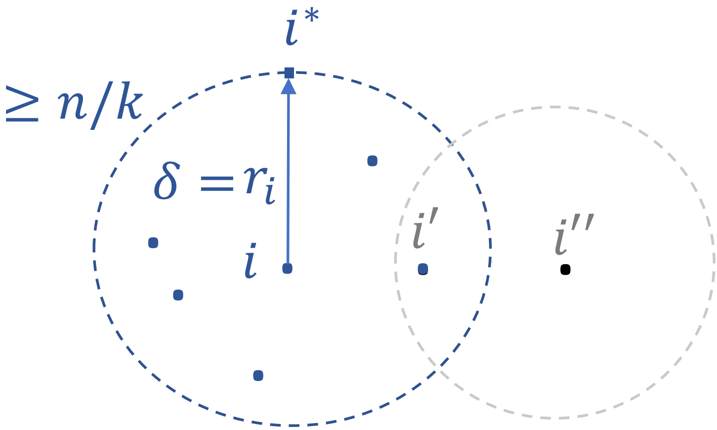


Individual Fairness

- Theorem [Jung et al. '19]: Greedy Capture returns a clustering solution that is 2-IF
- Proof:
- Let C be the solution that Greedy Capture returns
- Suppose for contradiction that some $i \in N$, $|B(i, r_i) \cap C| = 0$
- If $|B(i, r_i)| \geq n/k$, then i is included in the solution
- Otherwise, some of $i' \in B(i, r_i)$has been disregarded when it captured from a ball centered at i'' with radius at most r_i
- From triangle inequality, $d(i, i'') \leq d(i, i') + d(i', i'') \leq 2 \cdot r_i$



Core, JR and IF

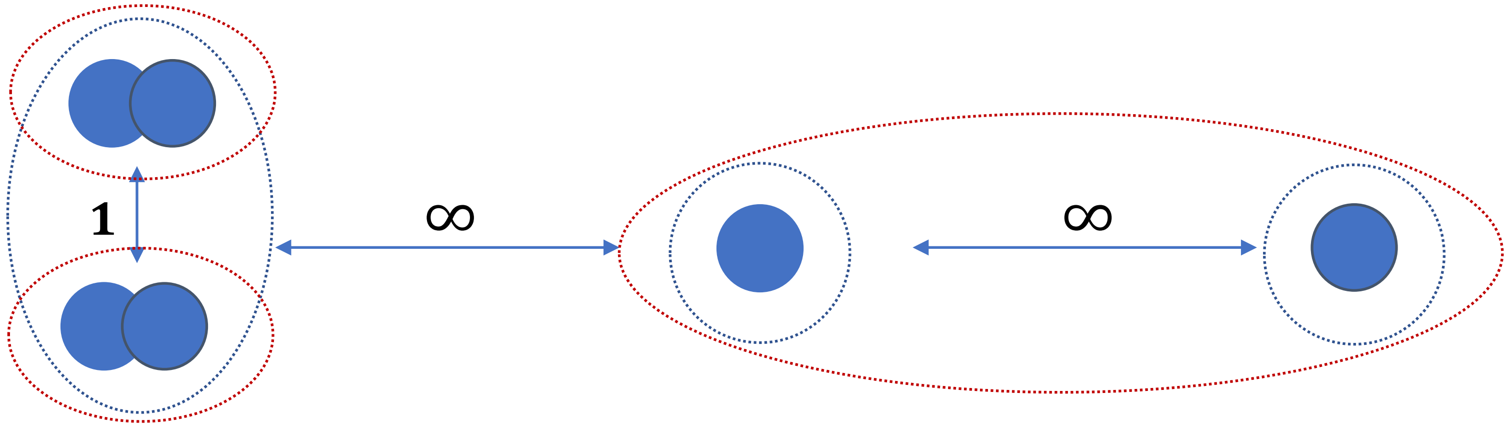
- Theorem: Greedy Capture returns a clustering solution that is JR, 2-IF and in the $1 + \sqrt{2}$-core .

- Theorem [Kellerhals and Peters '24]: Any clustering solution that satisfies JR, it also satisfies 2-IF and is in the $1 + \sqrt{2}$-core .

- Theorem [Kellerhals and Peters '24]:
  - ❑ Any clustering solution that satisfies α-IF, it is also in the $2 \cdot \alpha$-core
  - ❑ Any clustering solution that is in the α-core, it also satisfies $(1 + \alpha)$-IF

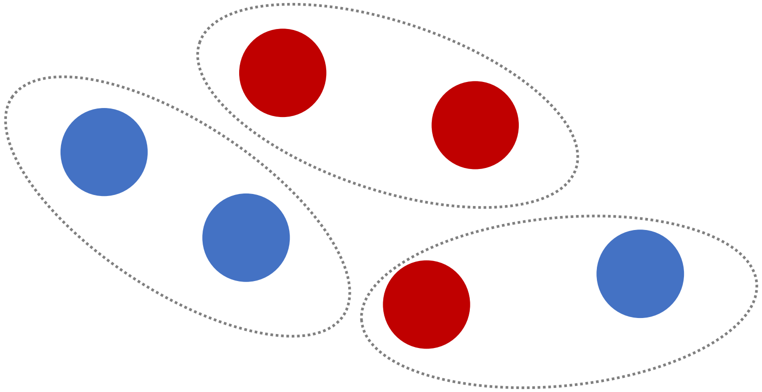
Core, JR and IF vs k-means, k-median, k-center

$k = 3$



Demographic Fairness

- Demographic Groups:
  - There is predefined set of protected groups (e.g. race or gender)
  - Each individual/data point belongs to one group
  - Disparate Impact in ML: The impact of a system across protected groups
  - Disparate Impact in Clustering: The impact on a group is measured by how many individuals of that group are in each cluster



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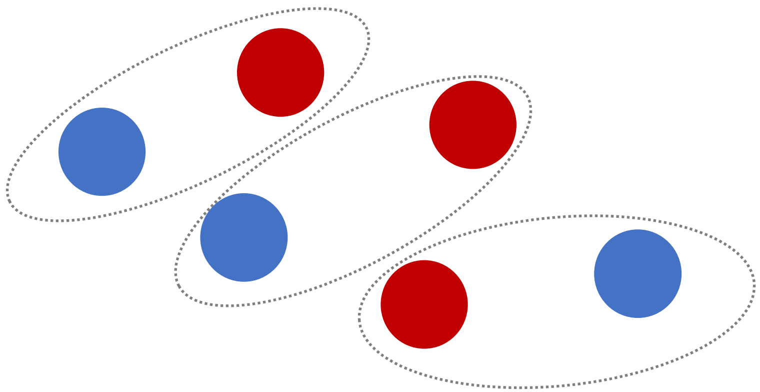

Balancedness

- Let $G_1, \dots, G_t$ be the protected groups
- Let $C = \{C_1, \dots, C_k\}$ be a clustering solution
- The balancedness in each cluster C_j is measured as:

$$balance(C_j) = \min_{i \neq i' \in [t]} \frac{|G_i \cap C_j|}{|G_{i'} \cap C_j|}$$

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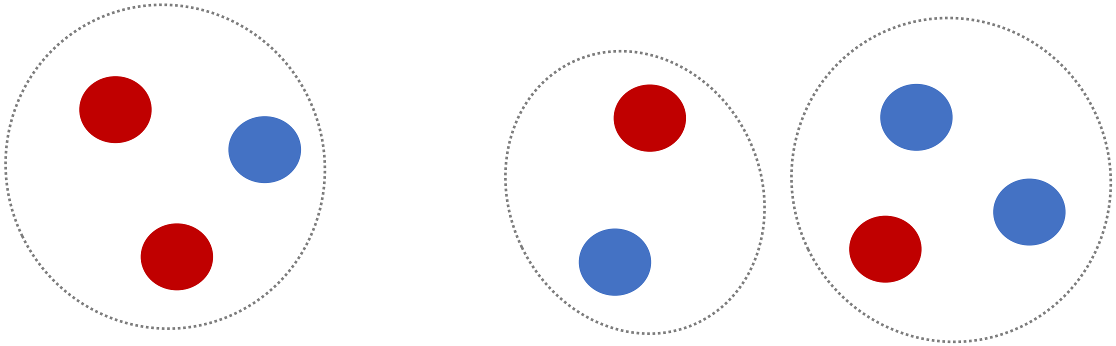

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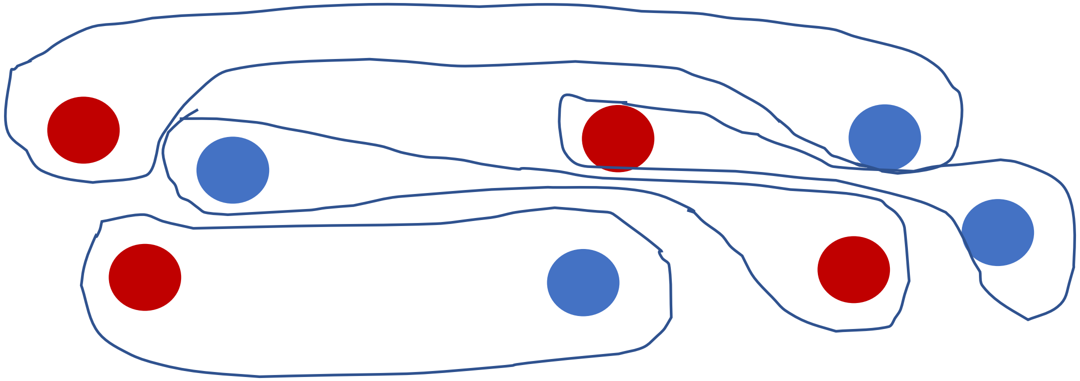

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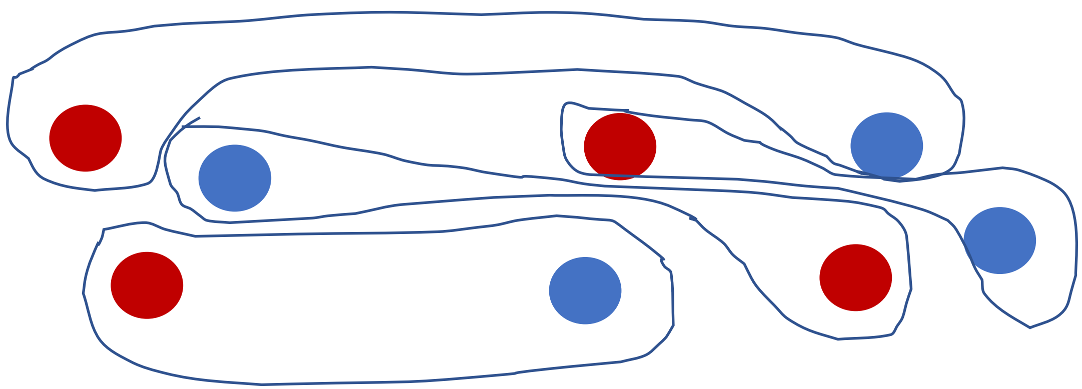

Bounded Representation

- Let $G_1, \dots, G_t$ be the protected groups
- Let $C = \{C_1, \dots, C_k\}$ be a clustering solution
- For (α, β)- bounded representation we require that

$$\alpha \leq \left|G_i \cap C_j\right| \leq \beta, \quad \forall i \in [t] \text{ and } \forall j \in [k]$$

- Standard objectives such as k-center, k-median and k-means are maximized subject to (α, β)- bounded representation constraints

- Open Question: Maximize the approximation to the core subject to (α, β)-bounded representation constraints