Fair Matching

Landscape

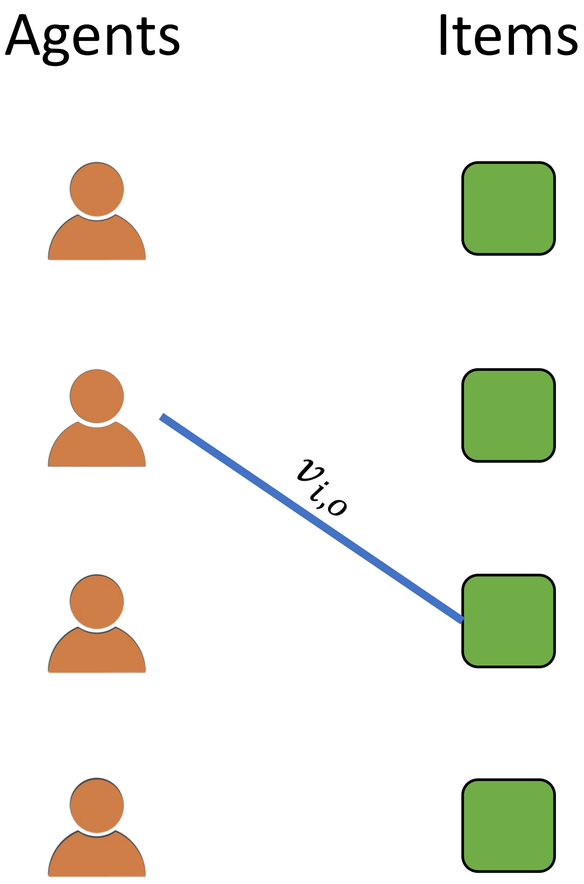
Matching	1-1	Many-Many
One-Sided	House Allocation	Course Allocation
Two-Sided	Marriage Problem	No standard name (But wide applications)

One-Sided, 1-1

House Allocation

- **Model**
  - Set of agents $N = \{1,2,\ldots,n\}$
  - Set of items M, $|M| = n$
  - Agent i has value $v_{i,o}$ for item $o \in M$

- **Matching**
  - $A_{i,o}$ = fraction of item o matched to agent i
  - $\sum_i A_{i,o} = 1, \forall o$, $\sum_o A_{i,o} = 1, \forall i$
    - "Constrained allocations"
  - $v_i(A_i) = \sum_o A_{i,o} \cdot v_{i,o}$



Integral Matchings

- $A_{i,o} \in \{0,1\} \Rightarrow$ integral perfect matching
  - Hard to provide non-trivial fairness guarantees (e.g., EF1 is vacuous)
  - Some agents will be happiest, some less so, some very unhappy

- Serial dictatorship
  - Define an ordering π over the agents
  - For $i = 1, \ldots, n$
    - Agent $\pi(i)$ picks her most favorite item from those still left
  - Satisfies PO (check!)

- What else can we do?
  - Maximize utilitarian welfare, Nash welfare, egalitarian welfare, …
  - Still just PO, but now rely on the exact utilities

Fractional Matchings

- $A_{i,o} \in [0,1] \Rightarrow$ fractional perfect matching
  - Can ask for non-trivial guarantees, e.g., EF+PO
- Competitive Equilibrium from Equal Incomes [HZ79]:
  - Need to set a price $p_o \geq 0$ for each item o
  - Notation: size $|A_i| = \sum_o A_{i,o}$, price $p(A_i) = \sum_o A_{i,o} \cdot p_o$
  - (A, p) is a CEEI if:
    - EI: $|A_i| \leq 1$ & $p(A_i) \leq 1$
    - CE: $v_i(A_i) \geq v_i(B_i)$ $\forall$ "feasible" B_i s.t. $|B_i| \leq 1$ & $p(B_i) \leq 1$
  - Existence: via Kakutani's fixed point theorem
  - Computation: PPAD-complete
  - EF (why?) + PO (why?)

Fair Division

- $A_{i,o} \in [0,1] \Rightarrow$ fractional allocation
  - Can ask for non-trivial guarantees, e.g., EF+PO

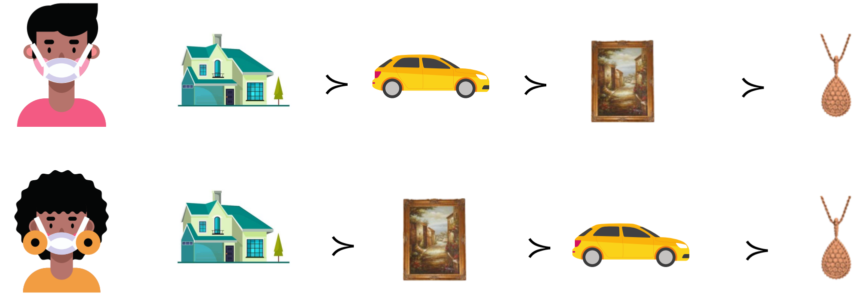
- Competitive Equilibrium from Equal Incomes:
  - Need to set a price $p_o \geq 0$ for each item o
  - Notation: ~~size $|A_i| = \sum_o A_{i,o}$~~, price $p(A_i) = \sum_o A_{i,o} \cdot p_o$
  - (A, p) is a CEEI if:
    - EI: ~~$|A_i| \leq 1$ &~~ $p(A_i) \leq 1$
    - CE: $v_i(A_i) \geq v_i(B_i)$ ∀ "feasible" B_i s.t. ~~$|B_i| \leq 1$ &~~ $p(B_i) \leq 1$

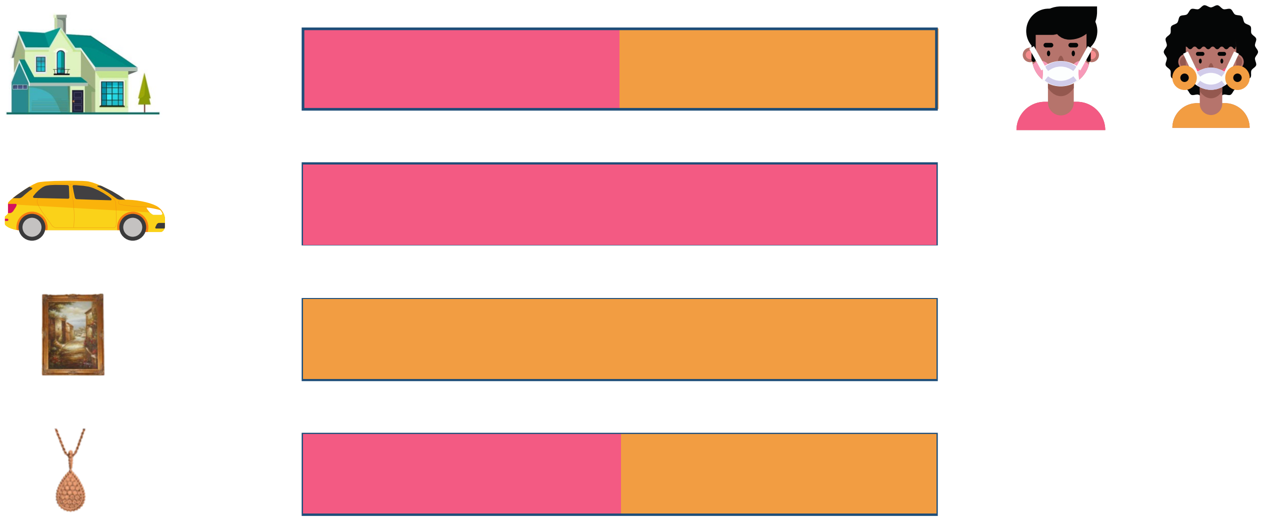
  - Known: (A, p) is CEEI iff A = a max Nash welfare allocation, p = its standard price measure: $p_o = \frac{v_{i,o}}{v_i(A_i)}$, where $A_{i,o} > 0$
    - Computable in strongly polynomial time

Fractional Matchings

- Probabilistic Serial
  - At time $t = 0$, each agent starts “eating” his most favorite item at the same rate of 1 item per unit time
  - As soon as an item is fully eaten up, all agents who were eating it shift to their respective next-best items

Probabilistic Serial





Envy-free

Probabilistic Serial

Probabilistic Serial

- Envy-free (Why?)
  - In fact, it is "SD-envy-free"
  - Because it achieves envy-freeness while using only the ordinal preferences, it is also envy-free with respect to all cardinal utilities that induce the same ordinal preferences

- Not Pareto optimal
  - But "SD-pareto-optimal": no other matching can be a Pareto improvement for all cardinal utilities that induce the same ordinal preferences

One-Sided,
many-to-many

Course Allocation

- Model
  - Set of agents $N = \{1,2,\dots,n\}$
  - Set of items M, $|M| = n$
  - Agent i has value $v_{i,o}$ for item $o \in M$

- Many-to-many Matching
  - $A_{i,o}$ = fraction of item o matched to agent i
  - $\sum_i A_{i,o} = d, \forall o, \sum_o A_{i,o} = d, \forall i$
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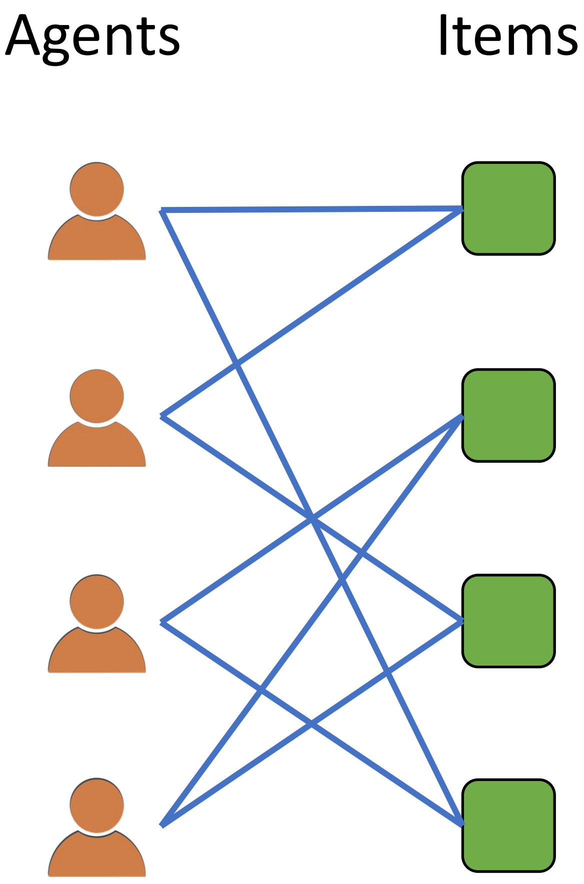


Course Allocation

- Now possible to seek non-trivial fairness guarantees even with integral many-to-many matchings

- Open question: Does there always exist an EF1+PO many-to-many matching?

- Recall that allocation of bads is a special case of this:
  - Allocate $n - 1$ copies of a good ("get out of doing chore c") for each chore c
  - Question remains open even for this special case

Two-Sided, 1-1

Stable Marriage

Stable Matching

- Recap Graph Theory:

- In graph $G = (V, E)$, a matching $M \subseteq E$ is a set of edges with no common vertices
  - That is, each vertex should have at most one incident edge
  - A matching is perfect if no vertex is left unmatched.

- G is a bipartite graph if there exist V_1, V_2 such that $V = V_1 \cup V_2$ and $E \subseteq V_1 \times V_2$

Stable Marriage Problem

- Bipartite graph, two sides with equal vertices
  - n men and n women (old school terminology ☹)
- Each man has a ranking over women & vice versa
  - E.g., Eden might prefer Alice $\succ$ Tina $\succ$ Maya
  - And Tina might prefer Tony $\succ$ Alan $\succ$ Eden
- Want: a perfect, stable matching
  - Match each man to a unique woman such that no pair of man m and woman w prefer each other to their current matches (such a pair is called a "blocking pair")

Example: Preferences

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

 $\succ$  $\succ$ 

Example: Matching 1

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

Question: Is this a stable matching?

Example: Matching 1

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

No, Albert and Emily form a **blocking pair**.

Example: Matching 2

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

Question: How about this matching?

Example: Matching 2

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

Yes! (Charles and Fergie are unhappy, but helpless.)

Does a stable matching always exist in the marriage problem?

Can we compute it efficiently?

Gale-Shapley 1962

- Men-Proposing Deferred Acceptance (MPDA):

1. Initially, no proposals, engagements, or matches are made.
2. While some man m is unengaged:
   - $w \leftarrow m$'s most preferred woman to whom m has not proposed yet
   - m proposes to w
   - If w is unengaged:
     - m and w are engaged
   - Else if w prefers m to her current partner m'
     - m and w are engaged, m' becomes unengaged
   - Else: w rejects m
3. Match all engaged pairs.

Example: MPDA

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

= proposed   = engaged   = rejected

Running Time

- Theorem: DA terminates in polynomial time (at most n^2 iterations of the outer loop)

- Proof:
  - In each iteration, a man proposes to someone to whom he has never proposed before.
  - n men, n women $\rightarrow$ $n \times n$ possible proposals
  - Can actually tighten a bit to $n(n-1)+1$ iterations

Matching

- Theorem: DA returns a perfect matching upon termination
- Proof:
  - Suppose it doesn't
  - Since there are an equal number of men and women, there must be a man m and a woman w who are both unengaged at the end
  - A woman becomes engaged at the first proposal and stays engaged
    - Hence, w must have never received a proposal
    - Hence, m never proposed to w
    - Hence, the algorithm can continue with m proposing to w
    - Contradiction!

Stable Matching

- Theorem: DA returns a stable matching

- Proof by contradiction:
  - Assume (m, w) is a blocking pair.

  - Case 1: m never proposed to w
    - m cannot be unmatched o/w algorithm would not terminate.
    - Men propose in the order of preference.
    - Hence, m must be matched with a woman he prefers to w
    - (m, w) is not a blocking pair

Stable Matching

- Theorem: DA returns a stable matching

- Proof by contradiction:
  - Assume (m, w) is a blocking pair.

  - Case 2: m proposed to w
    - w must have rejected m at some point
    - Women only reject to get better partners
    - At the end, w must be matched to a partner she prefers to m
    - (m, w) is not a blocking pair

Men-Optimal Stable Matching

- The stable matching found by MPDA is special.
- Valid partner: For a man m, call a woman w a valid partner if (m, w) is in *some* stable matching.
- Best valid partner: For a man m, a woman w is the best valid partner if she is a valid partner, and m prefers her to every other valid partner.
  - Denote the best valid partner of m by $best(m)$.

Men-Optimal Stable Matching

- Theorem: Every execution of MPDA returns the "men-optimal" stable matching: every man is matched to his best valid partner.
  - Surprising that this is a matching. E.g., it means two men cannot have the same best valid partner!

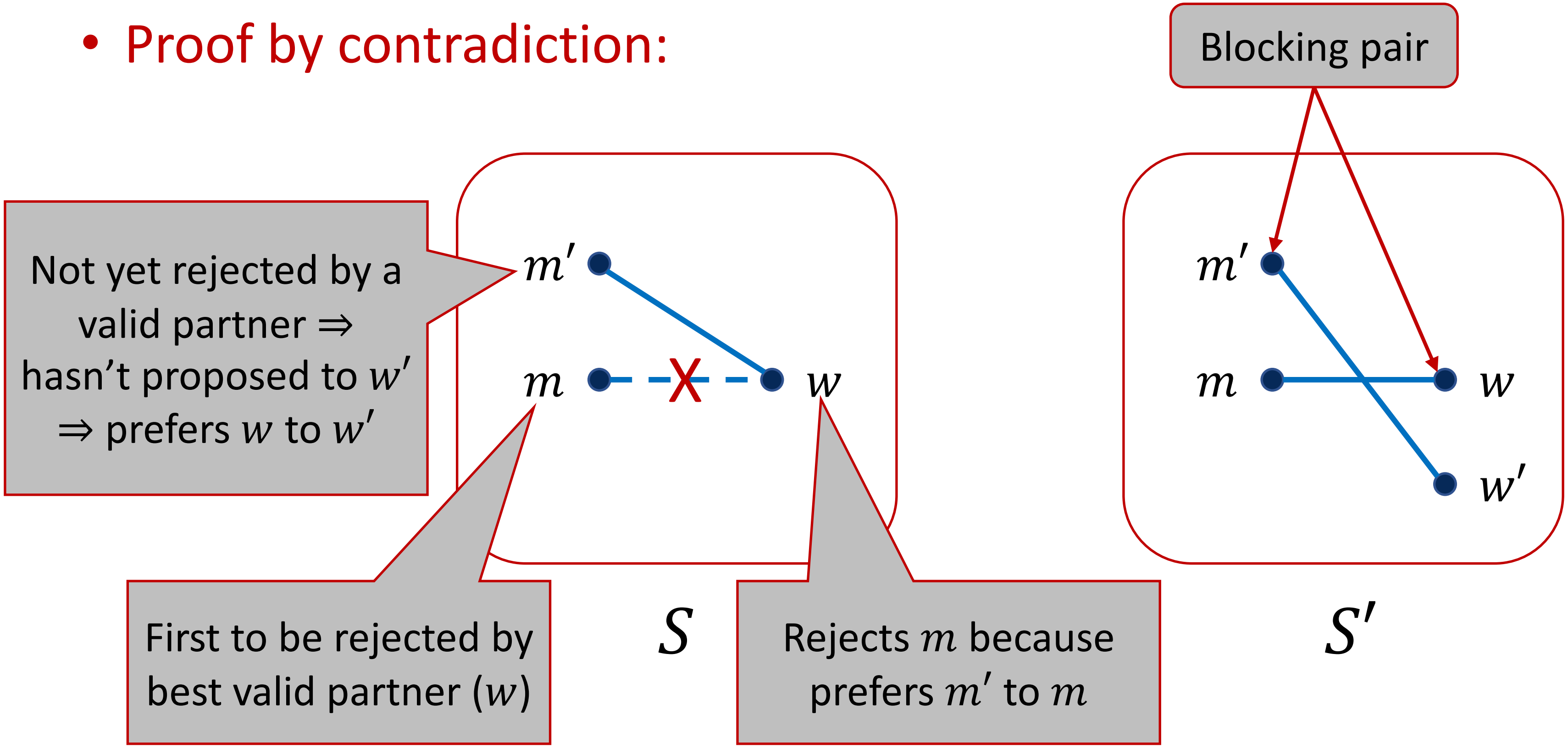
- Theorem: Every execution of MPDA produces the "women-pessimal" stable matching: every woman is matched to her worst valid partner.

Men-Optimal Stable Matching

- Theorem: Every execution of MPDA returns the men-optimal stable matching.
- Proof by contradiction:
  - Let S = matching returned by MPDA.
  - $m \leftarrow$ first man rejected by $best(m) = w$
  - $m' \leftarrow$ the more preferred man due to which w rejected m
  - w is valid for m, so (m, w) part of stable matching S'
  - $w' \leftarrow$ woman m' is matched to in S'
  - We show that S' cannot be stable because (m', w) is a blocking pair.

Men-Optimal Stable Matching

- Theorem: Every execution of MPDA returns the men-optimal stable matching.
- Proof by contradiction:



Strategyproofness

- Strategyproofness
  - An algorithm is called strategyproof if no agent can misrepresent her preferences to strictly improve her outcome in any instance.

- Theorem: MPDA is strategyproof for men.
  - We'll skip the proof of this.
  - Actually, it is "group-strategyproof".

- But the women might gain by misreporting.

- Theorem: No algorithm for the stable matching problem is strategyproof for both men and women.

Women-Proposing Version

- Women-Proposing Deferred Acceptance (WPDA)
  - Just flip the roles of men and women
  - Strategyproof for women, not strategyproof for men
  - Returns the women-optimal and men-pessimal stable matching

Extensions

- Unacceptable matches
  - Allow every agent to report a partial ranking
  - If woman w does not include man m in her preference list, it means she would rather be unmatched than matched with m. And vice versa.
  - (m, w) is blocking if each prefers the other over their current state (matched with another partner or unmatched)
  - Just m (or just w) can also be blocking if they prefer being unmatched than be matched to their current partner

- Magically, DA still produces a stable matching.

Extensions

- Resident Matching (or College Admission)
  - Men → residents (or students)
  - Women → hospitals (or colleges)
  - Each side has a ranked preference over the other side
  - But each hospital (or college) q can accept $c_q > 1$ residents (or students)
  - Many-to-one matching
- An extension of Deferred Acceptance works
  - Resident-proposing (resp. hospital-proposing) results in resident-optimal (resp. hospital-optimal) stable matching

Extensions

- For ~20 years, most people thought that these problems are very similar to the stable marriage problem

- Roth [1985] shows:
  - No stable matching algorithm is strategyproof for hospitals (or colleges).

Extensions

- Roommate Matching
  - Still one-to-one matching
  - But no partition into men and women
    - "Generalizing from bipartite graphs to general graphs"
  - Each of n agents submits a ranking over the other $n-1$ agents

- Unfortunately, there are instances where no stable matching exist.
  - A variant of DA can still find a stable matching *if* it exists.
  - Due to Irving [1985]

NRMP: Matching in Practice

- 1940s: Decentralized resident-hospital matching
  - Markets “unralveled”, offers came earlier and earlier, quality of matches decreased
- 1950s: NRMP introduces centralized “clearinghouse”
- 1960s: Gale-Shapley introduce DA
- 1984: Al Roth studies NRMP algorithm, finds it is really a version of DA!
- 1970s: Couples increasingly don’t use NRMP
- 1998: NRMP implements matching with couple constraints
  (stable matchings may not exist anymore…)
- More recently, DA applied to college admissions

Two-Sided, 1-1 revisited

Stability vs Envy-Freeness

- Stability vs EF
  - EF counts all agents as equals
  - Stability gives priority to those agents who are highly valued by agents on the other side
  - Each may be useful in different applications

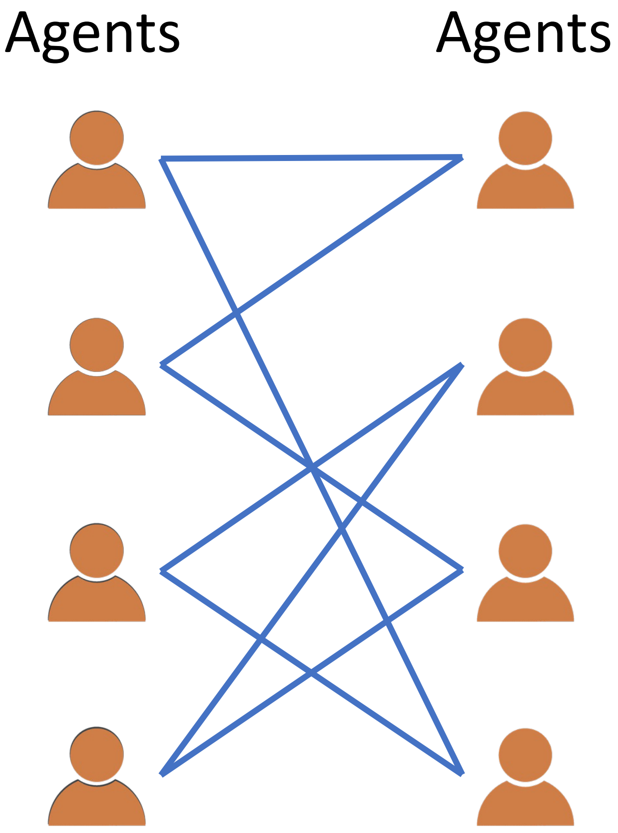
- Two-sided fractional perfect matchings
  - Can we get EF+PO?
  - Recall: For one-sided markets, this was possible [HZ79]
  - The answer is NO [TV23]
    - Counterexamples with asymmetric {0,1} values and symmetric {0,1,2} values
    - Open for symmetric {0,1} values

Two-Sided, many-to-many

Two-Sided, Many-to-Many

- Model
  - Sets of agents U, V ($|U| = |V| = n$)
  - Each $i \in U$ has $v_{i,j}$ for each $j \in V$
  - Each $j \in V$ has $v_{j,i}$ for each $i \in U$

- Many-to-many Matching
  - $A_{i,j}$ = fraction at which i matched to j
  - $\sum_{i \in U} A_{i,j} = d, \forall j \in V, \sum_{j \in V} A_{i,j} = d, \forall i \in U$
  - $v_i(A_i) = \sum_{j \in V} A_{i,j} \cdot v_{i,j}$
  - $v_j(A_j) = \sum_{i \in U} A_{i,j} \cdot v_{i,j}$



Doubly EF1 Matchings

- [Freeman et al, '20]
  - Matching A is doubly EF1 if no agent on either side envies another agent on their own side up to one of their matches
  - "EF1 among agents on the left, EF1 among agents on the right"

- Open question: Do doubly EF1 matchings always exist?

- Theorem: When agents on each side have the same ranking over the agents on the other side, then it does.