Proportional Representation in Voting

Credit: Dominik Peters' Wonderful Tutorial

Voting

- Set of n agents $N = \{1, \dots, n\}$
- Set of m candidates M
- Votes
  - Cardinal utilities $u_i: M \rightarrow \mathbb{R}_{\geq 0}$ (less prominent)
  - Ranked ballots $\succ_i$ (e.g., $a \succ_i b \succ_i c$)
  - Approval ballots $A_i \subseteq M$
    - Equivalent to binary cardinal utilities $c \in A_i \Leftrightarrow u_i(c) = 1$
- Goal
  - Single-winner voting: choose $c^* \in M$
  - Multiwinner voting: choose $S \subseteq M$ with $|S| \leq k$ (for given k)

“ABC” Voting

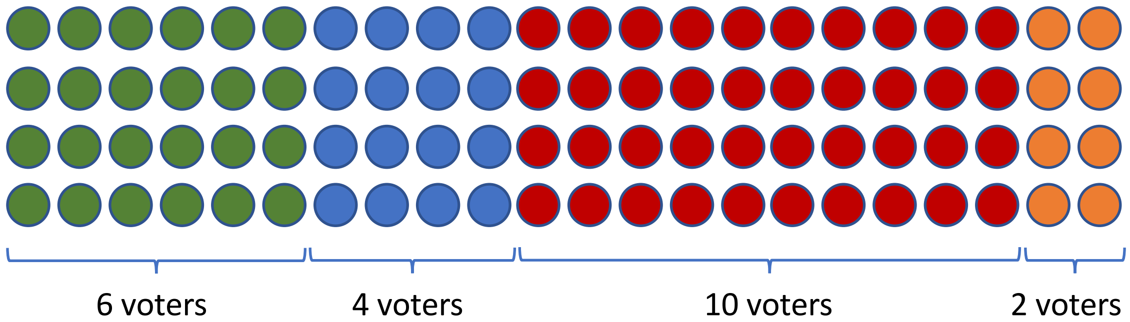
- Fairness
  - Difficult to define non-trivial fairness notions for single-winner voting
    - Can’t give each individual/group “proportionally deserved” utility
  - Much more interesting for multiwinner voting
    - We’ll focus on approval ballots, but many of the notions we’ll see have been extended to ranked ballots and cardinal utilities

- Approval-Based Multiwinner Voting
  - Each voter i approves a subset of candidates $A_i \subseteq M$
  - A subset of candidates $W \subseteq M, |W| \leq k$ is selected
  - Each voter i gets utility $u_i(W) = |W \cap A_i|$

Prominent Rules

- Thiele's Methods [1895]
  - Given a sequence $s = (s_1, s_2, \dots)$, select a committee W that maximizes $\sum_{i \in N} s_{u_i(W)}$
- Examples
  - Approval voting (AV): $s = (1,1,1, \dots)$
    - Selects the k candidates with the highest total approvals
  - Chamberlin-Courant (CC): $s = (1,0,0, \dots)$
    - Maximizes the number of voters for whom at least one approved candidate is selected
  - Proportional approval voting (PAV): $s = (1, {}^1/_2, {}^1/_3, \dots)$
    - In between AV and CC, but why exactly harmonic scores?

Why Harmonic Numbers?



$k = 11$

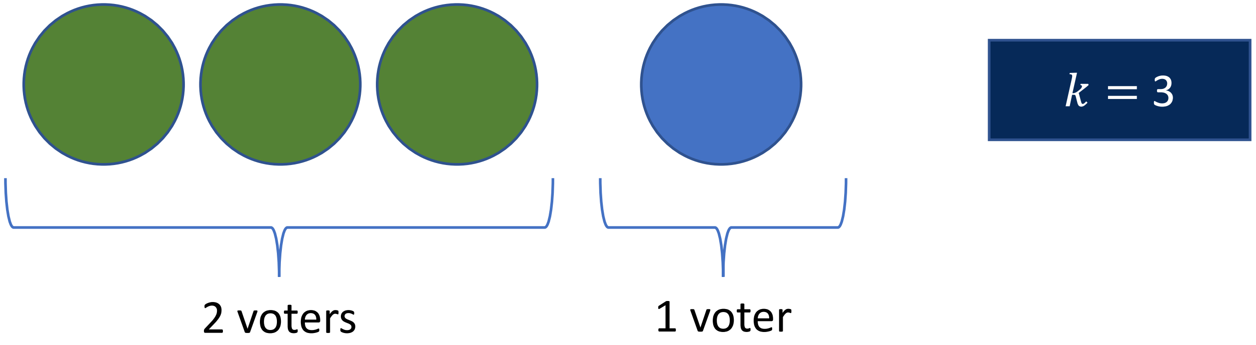
- "Proportionality"
  - We should select 3 , 2 , 5 , 1

Party-List PR

- Party-list instances
  - For all $i, j \in N$: either $A_i = A_j$ or $A_i \cap A_j = \emptyset$
  - For all $i \in N$: $|A_i| \geq k$
- Lower quota for party-list instances
  - For every party-list instance, $u_i(W) \geq \left\lfloor k \cdot {}^{n_i}/_{n} \right\rfloor$ for all $i \in N$, where $n_i = \left|\{j \in N : A_j = A_i\}\right|$
- AV, CC violate lower quota for party-list instances
- PAV satisfies it

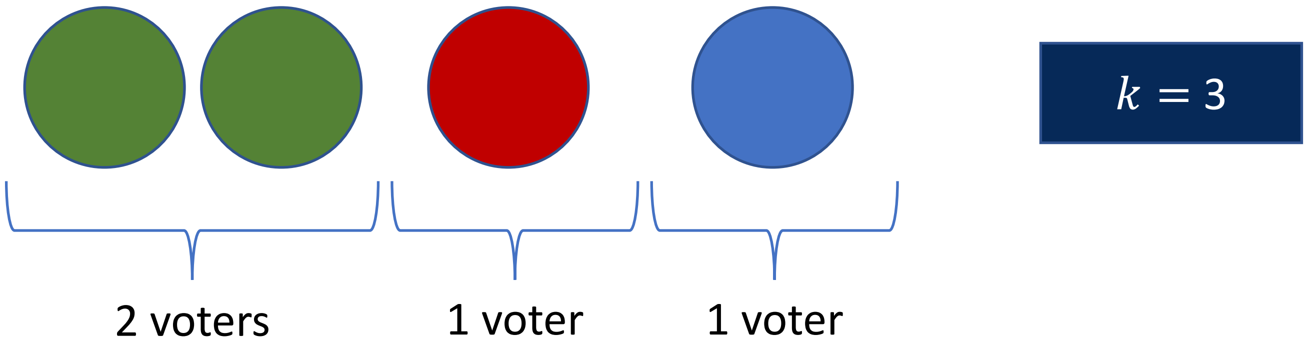
Party-List PR

- AV violates lower quota for party-list instances
  - 4 candidates $\{a, b, c, d\}$, $k = 3$
  - 2 voters approve $\{a, b, c\}$ and 1 voter approves d

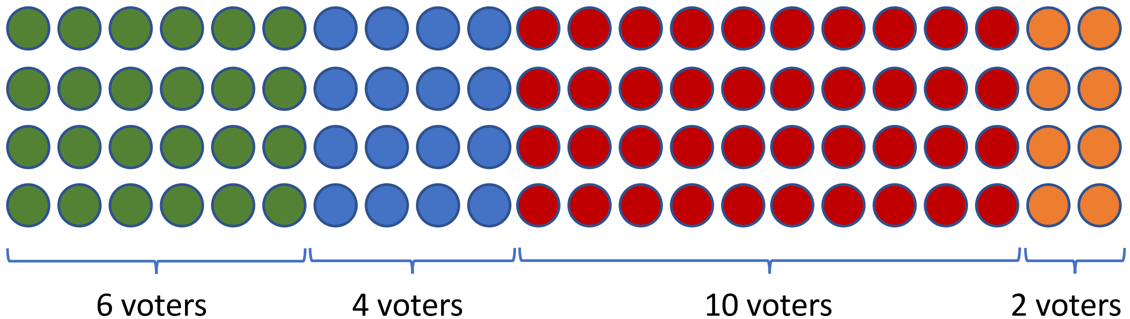


Party-List PR

- CC violates lower quota for party-list instances
  - 6 candidates $\{a, b, c, d\}, k = 3$
  - 2 voters approve $\{a, b\}$, 1 voter approves $\{c\}$, 1 voter approves $\{d\}$



Intuition Behind PAV



$k = 11$

- Party-list PR
  - We should select 3 , 2 , 5 , 1
  - PAV would have the desired result because:
    - 3rd , 2nd , 5th , 1st have the same marginal contribution = 2
    - We'll see a formal proof of PAV satisfying something stronger later
    - PAV known to be the only Thiele's method (and subject to additional axioms the only ABC rule) achieving this

Fairness for General Instances

- Issues
  - No well-separated "groups" of voters
  - A subset of voters may not be "fully cohesive" (having identical approval sets)

- We want to provide a utility guarantee to
  - …every possible subset (group) of voters that is…
  - …sufficiently large and cohesive and…
  - …their guarantee scales with their size and cohesiveness

Justified Representation (JR)

- Definition: W satisfies JR if
  - For all $S \subseteq N$
  - If $|S| \geq {}^{n}/_{k}$ (large) and $\left|\bigcap_{i \in S} A_i\right| \geq 1$ (cohesive)
  - Then $u_i(W) \geq 1$ for some $i \in S$
  - "If a group deserves one candidate and has a commonly approved candidate, then not every member should get 0 utility"

- Incomparable to party-list PR
- AV fails JR, CC and PAV satisfy JR

CC ⇒ JR

- Suppose CC selects W, which violates JR

- Then, there is a group $S \subseteq N$ such that
  - $|S| \geq {}^{n}/_{k}$
  - No $i \in S$ is "covered" ($u_i(W) = 0\ \forall i \in S$)
  - There is a candidate $c^* \in \cap_i A_i$

- Since W covers less than n voters in total, some $c \in W$ covers (is approved by) less than ${}^{n}/_{k}$ voters

- Replacing c with c^* gives a new committee that covers strictly more voters, a contradiction to W already maximizing this metric!

Extended Justified Representation (EJR)

- Definition: W satisfies EJR if
  - For all $S \subseteq N$ and $\ell \in \{1, \dots, k\}$
  - If $|S| \geq \ell \cdot {}^{n}/_{k}$ (large) and $|\cap_{i \in S} A_i| \geq \ell$ (cohesive)
  - Then $u_i(W) \geq \ell$ for some $i \in S$
  - "If a group deserves ℓ candidates and has ℓ commonly approved candidates, then not every member should get less than ℓ utility"
  - JR imposes this but only for $\ell = 1$, so EJR $\Rightarrow$ JR

- AV and CC fail EJR, PAV satisfies it

PAV $\Rightarrow$ EJR

- Suppose PAV selects W, which violates EJR
  - $PAV(W) = \sum_{i \in N} \frac{1}{u_i(W)}$
- Then, there is a group $S \subseteq N$ and $\ell \in \{1, \dots, k\}$ such that
  - $|S| \geq \ell \cdot {}^{n}/_{k}$
  - $u_i(W) < \ell, \quad \forall i \in S$
  - $|\cap_i A_i| \geq \ell \Rightarrow$ there exists $c^* \in \cap_i A_i \setminus W$ (Why?)
- Consider $\widetilde{W} = W \cup \{c^*\}$
  - $PAV\left(\widetilde{W}\right) \geq PAV(W) + |S| \cdot \frac{1}{\ell} \geq PAV(W) + \frac{n}{k}$
- Claim: Can remove some $c \in \widetilde{W}$ and lower score by $< \frac{n}{k}$

PAV $\Rightarrow$ EJR

- Claim: Can remove some $c \in \widetilde{W}$ and lower score by $< \frac{n}{k}$
- Proof:
  - Suffices to prove that average reduction across $c \in \widetilde{W}$ is less than $\frac{n}{k}$
  - Reduction when removing $c \in \widetilde{W} = \sum_{i:c\in A_i} \frac{1}{u_i(\widetilde{W})}$
  - Average reduction:

$$\frac{1}{k+1} \cdot \sum_{c\in\widetilde{W}} \sum_{i:c\in A_i} \frac{1}{u_i(\widetilde{W})} = \frac{1}{k+1} \cdot \sum_{i\in N} \sum_{c\in A_i\cap\widetilde{W}} \frac{1}{u_i(\widetilde{W})}$$

$$= \frac{1}{k+1} \cdot \sum_{i\in N} 1$$

$$= \frac{n}{k+1} < \frac{n}{k}$$

Computation of PAV

- Computing PAV is NP-complete
- What about a greedy approximation?
  - Sequential PAV
    - $W \leftarrow \emptyset$
    - **while** $|W| < k$ **do**
      - Find c which maximizes $PAV(W \cup \{c\})$
      - $W \leftarrow W \cup \{c\}$
  - Achieves at least $\left(1 - \frac{1}{e}\right)$ fraction of optimal PAV score
    - PAV score is a submodular function
  - But fails to satisfy EJR

Computation of PAV

- In practice, exact PAV solution can be computed via a BILP

- Binary variables:
  - $y_c \rightarrow$ Is candidate c selected?
  - $x_{i,\ell} \rightarrow$ Is $u_i(\{c : y_c = 1\}) \geq \ell$?

- Maximize $\sum_{i \in N} \sum_{\ell=1}^{k} \frac{1}{\ell} \cdot x_{i,\ell}$

  subject to $\sum_{\ell=1}^{k} x_{i,\ell} = \sum_{c \in A_i} y_c$ for all i

  $\sum_c y_c = k$

  $y_c, x_{i,\ell} \in \{0,1\}$ for all i, ℓ, c

$\leftarrow$ Why does this work?

Fully Justified Representation (FJR)

- **Definition:** W satisfies FJR if
  - For all $S \subseteq N, T \subseteq M$ and $\ell, \beta \in \{1, \dots, k\}$
  - If $|S| \geq |T| \cdot {}^{n}/_{k}$ (large) and $u_i(T) \geq \beta, \forall i \in S$ (cohesive)
  - Then $u_i(W) \geq \beta$ for some $i \in S$
  - "If a group deserves ℓ candidates and can propose a set of ℓ candidates from which each member gets at least β utility, then not every member should get less than β utility"
  - EJR imposes this but only for $\beta = \ell$, which would imply $T \subseteq \cap_{i \in S} A_i$, so we just wrote $|\cap_{i \in S} A_i| \geq \ell$
  - FJR $\Rightarrow$ EJR

- **Bad news:** PAV (and every other known "natural" rule) violates FJR

Fully Justified Representation (FJR)

- FJR is satisfiable via a simple polynomial-time greedy rule

- Greedy Cohesive Rule (GCR):
  - $W \leftarrow \emptyset$
  - $N^a \leftarrow N$ ("active voters")
  - **while** $\exists \beta > 0,\ S \subseteq N^a,\ T \subseteq M \setminus W$
    s.t. $|S| \geq |T| \cdot \frac{n}{k}$ and $\min_{i \in S} u_i(T) > \beta$ **do**
    - Pick such (β, S, T) with the highest β (break ties arbitrarily)
    - $W \leftarrow W \cup T, N^a \leftarrow N^a \setminus S$
  - **return** W

- Greedily find the most cohesive group of voters and add their suggested group of candidates

(Weak) Core

- Definition: W satisfies core if
  - For all $S \subseteq N$ and $T \subseteq M$
  - If $|S| \geq |T| \cdot {}^{n}/_{k}$ (large)
  - Then $u_i(W) \geq u_i(T)$ for some $i \in S$
  - "If a group can afford T, then T should not be a (strict) Pareto improvement for the group"
  - FJR only imposes $\max_{i \in S} u_i(W) \geq \min_{i \in S} u_i(T)$, so core ⇒ FJR

- **Major open question**
  - For ABC voting, does there always exist a committee in the core?

Notes

- Other fairness definitions
  - EJR+, SJR, AJR, PJR, PRJ+, UJR, CS, proportionality degree, …
  - See [Justified Representation wiki](#) for more details

Participatory Budgeting

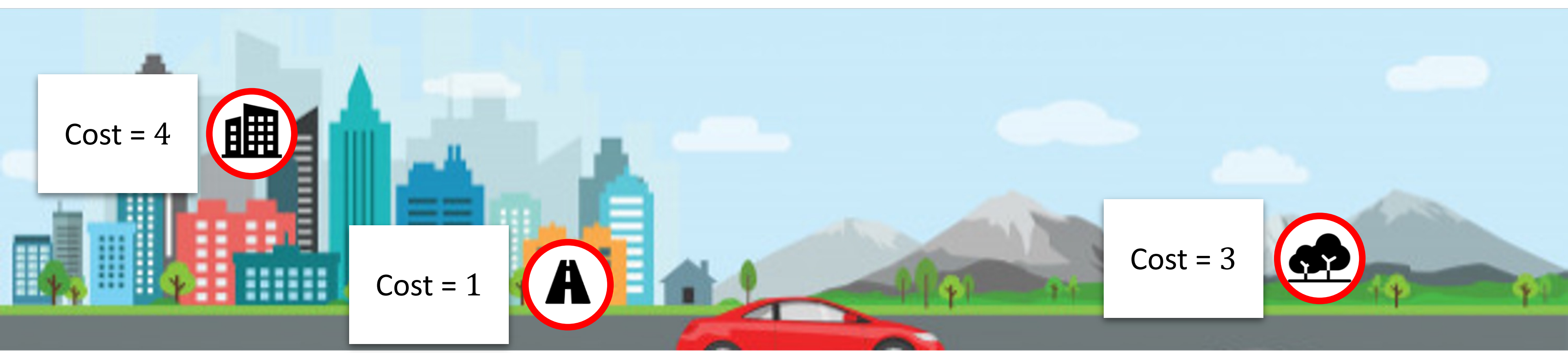
- Set of n agents $N = \{1, \dots, n\}$

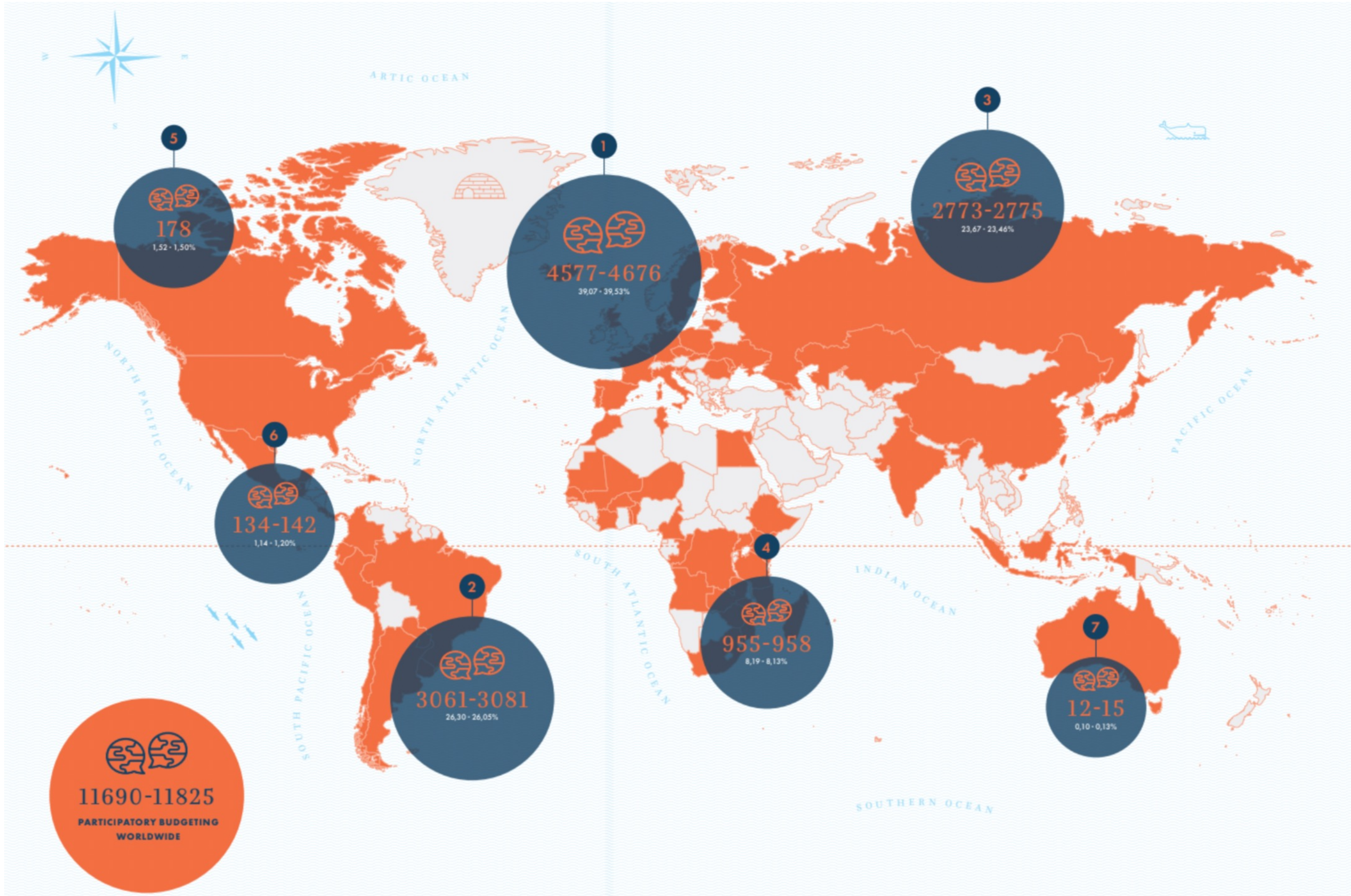
- Set of m projects M
  - Each project $a \in M$ has a cost c_a
  - Total budget is B

- Votes: cardinal utilities $u_i : M \to \mathbb{R}_{\geq 0}$
  - Other ballot formats also commonly studied (and more prevalent)

- Goal: choose $W \subseteq M$ with $c(W) \triangleq \sum_{a \in W} c_a \leq B$
  - Generalization of multiwinner voting

	7	3	2
	3	1	1
	1	2	5





The World Atlas of Participatory Budgeting, 2019

Method of Equal Shares

- A new method [Peters & Skowron ‘20]
- For multiwinner voting
  - Satisfies EJR *and* is polynomial-time computable
  - Recall: PAV is NP-hard to compute
- Extends to participatory budgeting
  - Satisfies a slight relaxation of EJR
  - EJR is satisfiable but not by any polynomial-time rule (unless P=NP)
- Has already been used by several cities
- In-depth explanation at https://equalshares.net/