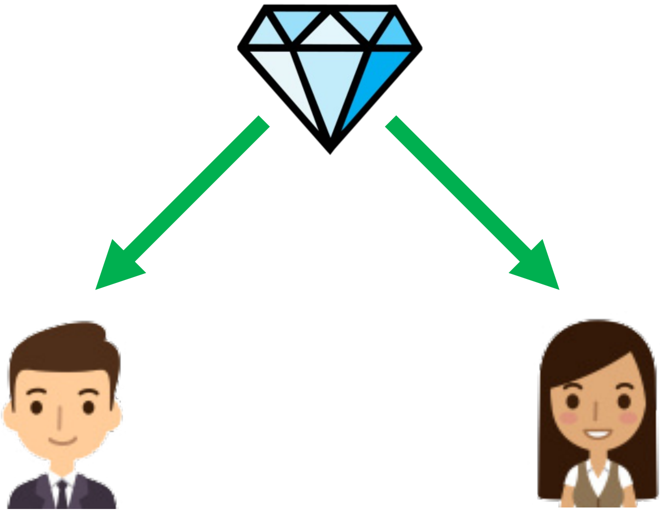
Fair Allocation 2: Indivisible Resources

Indivisible Goods

- Goods which cannot be shared among players
  - E.g., house, painting, car, jewelry, …
- Problem: Envy-free allocations may not exist!



Model

- Set of n agents $N = \{1, \dots, n\}$
- Set of m indivisible goods M
- Valuation function of agent i is $V_i: 2^M \to \mathbb{R}_{\geq 0}$
  - Additive: $V_i(S) = \sum_{g \in S} V_i(\{g\})$
  - We write $v_{i,g}$ to denote $V_i(\{g\})$ for simplicity
- Allocation $A = (A_1, \dots, A_m)$ is a partition of M
  - $\cup_i A_i = M$ and $A_i \cap A_j = \emptyset, \forall i, j$
  - For *partial* allocations, we drop the $\cup_i A_i = M$ requirement

Indivisible Goods

				
	8	7	20	5
	9	11	12	8
	9	10	18	3

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EF1

- Envy-freeness up to one good (EF1):

$$\forall i, j \in N, \exists g \in A_j : V_i(A_i) \geq V_i\left(A_j \setminus \{g\}\right)$$

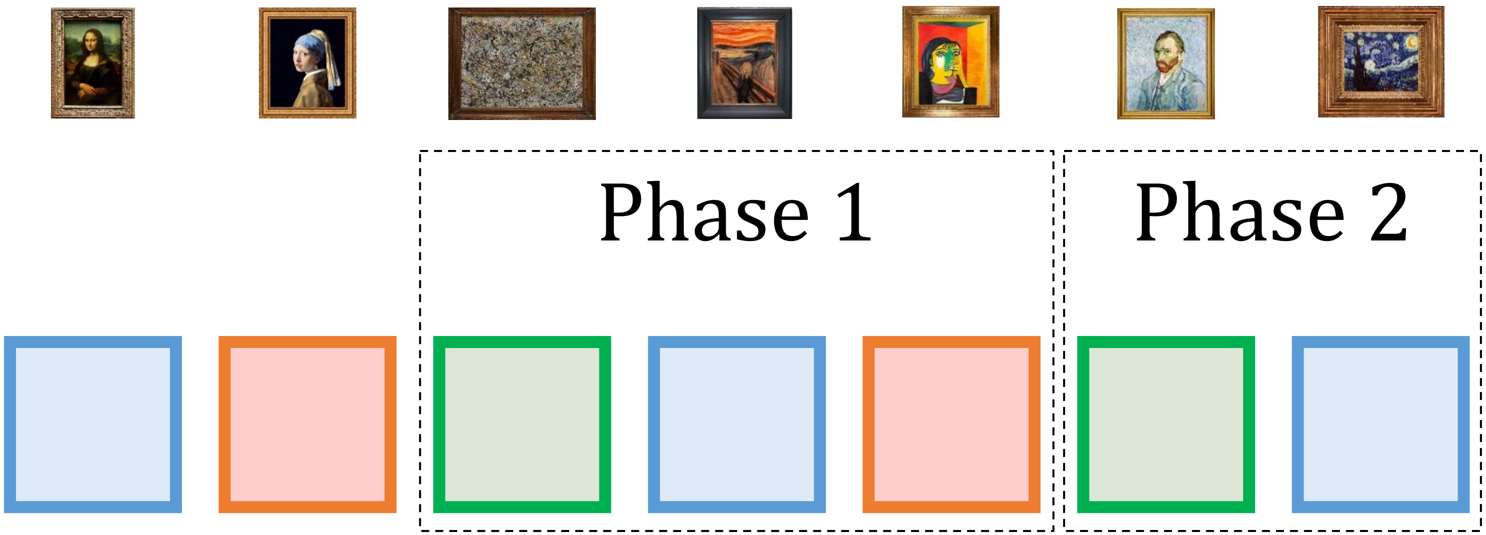
  - Technically, we need either this or $A_j = \emptyset$.

- In words…
  - "If i envies j, there must be some good in j's bundle such that removing it would make i envy-free of j."

- Question: Does there always exist an EF1 allocation?

EF1

- Yes, a simple round-robin procedure guarantees EF1
  - Order the agents arbitrarily (say $1,2,\dots,n$)
  - In a cyclic fashion, agents arrive one-by-one and pick the item they like the most among the ones left



EF1 + PO

- Pareto optimality (PO)
  - An allocation A is Pareto optimal if there is no other allocation B such that $V_i(B_i) \geq V_i(A_i)$ for all i and the inequality is strict for at least one i
- Sadly, round-robin does not always return a PO allocation
  - There exist instances in which, by reallocating items at the end, we can make all agents strictly happier
- Question: Does there always exist an allocation that is both EF1 and PO simultaneously?

EF1+PO?

- Maximum Nash Welfare (MNW) to the rescue!
  - Essentially, maximize the Nash welfare across all integral allocations

- Theorem [Caragiannis et al. '16]
  - (Almost true) Any allocation in $\text{argmax}_A \prod_{i \in N} V_i(A_i)$ is EF1 + PO.
  - [Conitzer et al. '19] Actually, it satisfies "group fairness up to one", which is stronger than EF1.

EF1+PO?

What is wrong in these arguments?

- Proof that A maximizing $\prod_i v_i(A_i)$ is EF1 + PO
  - PO is obvious
    - Suppose for contradiction that there is an allocation B such that $V_i(B_i) \geq V_i(A_i)$ for each i and $V_i(B_i) > V_i(A_i)$ for at least one i
    - Then, $\prod_i V_i(B_i) \geq \prod_i V_i(A_i)$, which is a contradiction
  - EF1 requires a bit more work
    - Fix any agents i, j and consider moving good $g \in A_j$ to A_i
    - A is MNW $\Rightarrow$ $V_i(A_i \cup \{g\}) \cdot V_j(A_j \setminus \{g\}) \leq V_i(A_i) \cdot V_j(A_j)$
    - $1 - \frac{v_{j,g}}{V_j(A_j)} \leq 1 - \frac{v_{i,g}}{V_i(A_i \cup \{g\})} \leq 1 - \frac{v_{i,g}}{V_i(A_i \cup \{g^*\})} \Rightarrow \frac{v_{j,g}}{V_j(A_j)} \geq \frac{v_i(g)}{V_i(A_i \cup \{g^*\})}$
      - Here, $g^* \in A_j$ is the good liked the most by i
    - Summing over all $g \in A_j$, we get $v_i(A_i \cup \{g^*\}) \geq v_i(A_j)$, which means i doesn't envy j up to good g^*

EF1+PO?

- Edge case: all allocations have zero Nash welfare
  - E.g., allocate two goods between three agents
  - Allocating both goods to a single agent can violate EF1

  - Requires a slight modification of the rule in this edge case
    - Step 1: Choose a subset of agents $S \subseteq N$ with largest $|S|$ such that it is possible to give a positive utility to each agent in S simultaneously
    - Step 2: Choose $\text{argmax}_A \prod_{i \in S} V_i(A_i)$

  - Quick questions:
    - How does this fix the example above?
    - Why did we not need this subtlety for cake-cutting?
    - Does this theorem generalize the one for cake-cutting?

Computation

- For indivisible goods, finding an MNW allocation is strongly NP-hard (NP-hard even if all values are bounded)

- Open Question:
    - Can we compute *some* EF1+PO allocation in polynomial time?
    - [Barman et al., '17]:
        - There exists a pseudo-polynomial time algorithm for finding an EF1+PO allocation
            - Time is polynomial in n, m, and $\max_{i,g} v_{i,g}$
            - Already better than the time complexity of computing an MNW allocation

EFX

- Envy-freeness up to any good (EFX)
  - $\forall i, j \in N, \forall g \in A_j : V_i(A_i) \geq V_i(A_j \backslash \{g\})$
  - In words, i shouldn't envy j if she removes *any* good from j's bundle
  - EFX $\Rightarrow$ EF1 $\left(\forall i, j \in N, \exists g \in A_j : V_i(A_i) \geq V_i(A_j \backslash \{g\})\right)$
- EF1 vs EFX example:
  - {A → P1; B,C → P2} is EF1 but not EFX, whereas .
  - {A,B → P1; C → P2} is EFX.

	A	B	C
P1	5	1	10
P2	0	1	10

- Open question: Does there always exist EFX allocation?

EFX

- (Easy to prove) EFX allocation always exists when…
  - Agents have identical valuations (i.e. $V_i = V_j$ for all i, j)
  - Agents have binary valuations (i.e. $v_{i,g} \in \{0,1\}$ for all i, g)
  - There are $n = 2$ agents with general additive valuations

- But answering this question in general (or even in some other special cases) has proved to be surprisingly difficult!

EFX: Recent Progress

- Partial allocations
  - [Caragiannis et al., '19]: There exists a partial EFX allocation A that has at least half of the optimal Nash welfare
  - [Ray Chaudhury et al., '19]: There exists a partial EFX allocation A such that for the set of unallocated goods U, $|U| \leq n-1$ and $V_i(A_i) \geq V_i(U)$ for all i

- Restricted number of agents
  - [Ray Chaudhury et al., '20]: There exists a complete EFX allocation with $n=3$ agents

- Restricted valuations
  - [Amanatidis et al., '20]: Maximizing Nash welfare achieves EFX when there exist a, b such that $v_{i,g} \in \{a, b\}$ for all i, g

MMS

- Maximin Share Guarantee (MMS):
  - Generalization of “cut and choose” for n players
  - MMS value of agent i =
    - The highest value that agent i can get…
    - If *she* divides the goods into n bundles…
    - But receives the worst bundle according to her valuation
  - Let $\mathcal{P}_n(M)$ = family of partitions of M into n bundles

$$MMS_i = \max_{(B_1,\ldots,B_n)\in\mathcal{P}_n(M)} \min_{k\in\{1,\ldots,n\}} V_i(B_k).$$

  - Allocation A is α-MMS if $V_i(A_i) \geq \alpha \cdot MMS_i$ for all i

MMS

- [Procaccia & Wang, '14]: MMS impossible, $^2/_3$ - MMS exists
- [Amanatidis et al., '17]: $(^2/_3 - \epsilon)$ - MMS in polynomial time
- [Ghodsi et al. '17]: $^3/_4$ - MMS exists, $(^3/_4 - \epsilon)$ - MMS in polynomial time
- [Garg & Taki, '20]: $^3/_4$ - MMS in polynomial time, $(^3/_4 + ^1/_{12n})$ - MMS exists
- [Feige et al. '21]: $(^{39}/_{40} + \epsilon)$ - MMS impossible
- [Akrami et al. '23]: $(^3/_4 + \min(^1/_{36}, ^3/_{16n-4}))$ - MMS exists
- [Hosseini et al. '22]: 1-out-of-$^{3n}/_2$ MMS exists, computable in polynomial time
  - Agent hypothetically partitions goods into $3n/2$ (instead of n) bundles and gets the worst of them

- Open questions:
  - What is the best α-MMS approximation possible? Does 1-out-of-$(n+1)$ MMS always exist?

Allocating Bads

- Costs instead of utilities
  - $c_{i,b}$ = cost of player i for bad b
    - $C_i(S) = \sum_{b \in S} c_{i,b}$
  - EF: $\forall i, j \;\; C_i(A_i) \leq C_i(A_j)$
  - PO: There is no allocation B such that $C_i(B_i) \leq C_i(A_i)$ for all i and at least one inequality is strict

- Divisible bads
  - An EF + PO allocation always exists
  - However, we can no longer just maximize the product (of what?)
  - Open question: Can we compute an EF+PO allocation of divisible bads in polynomial time?

Allocating Bads

- Indivisible bads
  - EF1: $\forall i, j \;\; \exists b \in A_i \;\; C_i(A_i \setminus \{b\}) \le C_i(A_j)$
  - EFX: $\forall i, j \;\; \forall b \in A_i \; C_i(A_i \setminus \{b\}) \le C_i(A_j)$
  - Open Question 1:
    - Does there always exist an EF1 + PO allocation?
  - Open Question 2:
    - Does there always exist an EFX allocation?
  - Many more open problems for allocating bads

Randomization

- Can we randomize over (ex-post) fair allocations to achieve exact fairness ex-ante (in expectation)?
  - Ex-ante EF: $\mathbb{E}[V_i(A_i)] \geq \mathbb{E}[V_i(A_j)], \forall i, j$
  - Ex-ante Prop: $\mathbb{E}[V_i(A_i)] \geq {}^1/_n, \forall i$
  - Ex-post means the property must be satisfied by *every deterministic allocation in the support*

- Known results
  - [Freeman et al. '20]: Ex-ante EF + ex-post EF1
  - [Freeman et al. '20]: Ex-ante EF + Ex-ante PO + ex-post Prop1
  - [Babaioff et al. '22]: Ex-ante Prop + Ex-post (Prop1 + ${}^1/_2$-MMS)

- Open question: Ex-ante EF + Ex-post (EF1+PO)?