Fair Allocation 1: Divisible Resources

Credit for some of the illustrations: Ariel D. Procaccia

Cake-Cutting

- A heterogeneous divisible good
  - Heterogeneous = same part may be valued differently by different agents
  - Divisible = can be divided between agents

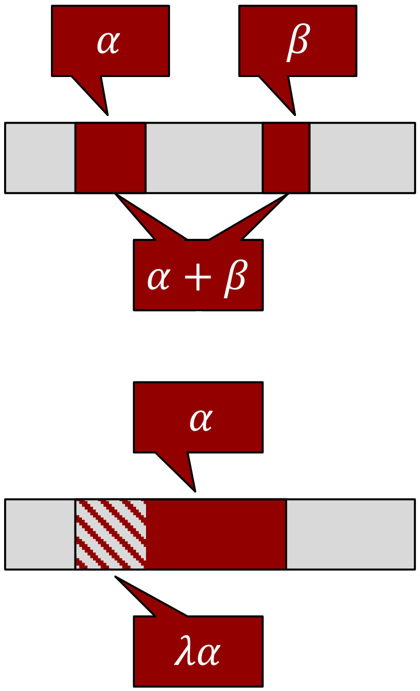
- Cake $C = [0,1]$
  - Almost without loss of generality
- Agents $N = \{1, \dots, n\}$
- Piece of cake $X \subseteq [0,1]$ = finite union of disjoint intervals
- Allocation $A = (A_1, \dots, A_n)$
  - Partition of the cake where each A_i is a piece of the cake

Agent Valuations

- Valuation of agent i is given by an integrable value density function $f_i: [0,1] \rightarrow \mathbb{R}_+$
  - Her value for a piece of cake X is $V_i(X) = \int_{x \in X} f_i(x) dx$

- Two key properties
  - Additive: For $X \cap Y = \emptyset$, $V_i(X) + V_i(Y) = V_i(X \cup Y)$
  - Divisible: $\forall \lambda \in [0,1]$ and X, $\exists Y \subseteq X$ s.t. $V_i(Y) = \lambda V_i(X)$

- WLOG
  - Normalized: $V_i([0,1]) = 1$



Fairness Goals

- What kind of fairness might we want from an allocation A?

- Proportionality (Prop):

$$\forall i \in N: V_i(A_i) \geq \frac{1}{n}$$

- Envy-Freeness (EF):

$$\forall i, j \in N: V_i(A_i) \geq V_i(A_j)$$

- Equitability (EQ):

$$\forall i, j \in N: V_i(A_i) = V_j(A_j)$$

Only makes sense with normalization

Fairness Goals

- Prop: $\forall i \in N: V_i(A_i) \geq 1/n$
- EF: $\forall i, j \in N: V_i(A_i) \geq V_i(A_j)$

- Question:
  What is the relation between proportionality and EF?

1. Prop ⇒ EF
2. EF ⇒ Prop
3. Equivalent
4. Incomparable

Cut-and-Choose

- Algorithm for $n = 2$ agents

- Agent 1 divides the cake into two pieces X, Y s.t.
$$V_1(X) = V_1(Y) = 1/2$$

- Agent 2 chooses the piece she prefers.

- This is EF and therefore proportional.
  - Why?

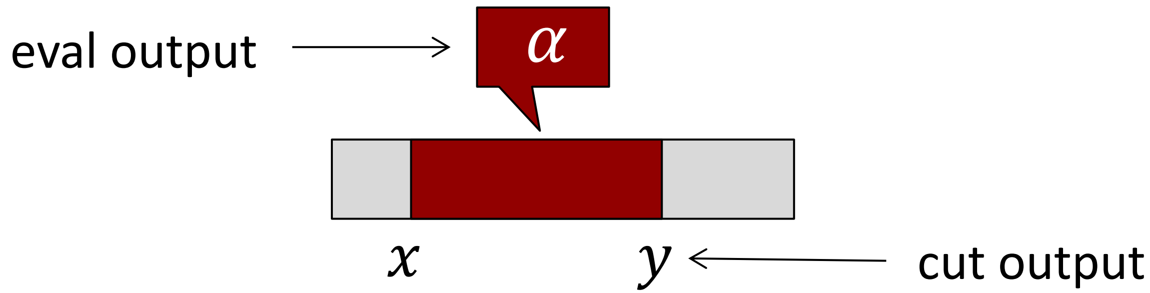
Measuring Complexity

- Running time does not make sense
  - Typically, we measure the running time as a function of the length of input encoded in binary
  - Our input consists of functions V_i, which requires infinitely many bits to encode
  - We want running time just as a function of n.

- Query models make sense
  - Allow specific types of queries to agents' valuation functions
  - Measure the number of queries that need to be made in order to find an allocation satisfying the given properties

Robertson-Webb Model

- Two types of queries to an agent's valuation function V_i
  - $\text{Eval}_i(x, y)$ returns $V_i([x, y])$
  - $\text{Cut}_i(x, \alpha)$ returns the smallest y such that $V_i([x, y]) = \alpha$
    - If no such y exists, then it returns 1



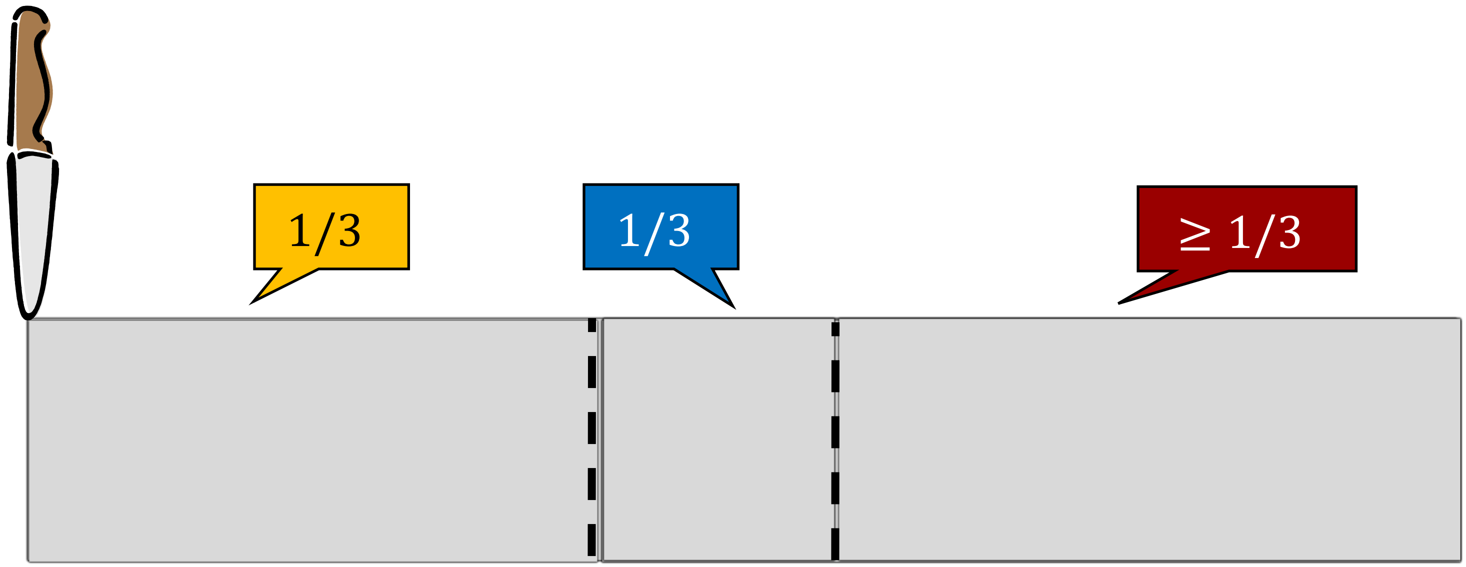
- Question:
  - How many queries are needed to find an EF allocation when $n = 2$?

Dubins-Spanier

- Protocol for finding a proportional allocation for n agents

- Referee starts with a knife at 0
- Referee continuously moves the knife to the right
- Repeat $n - 1$ times: Whenever the piece to the left of knife is worth $1/n$ to a agent, the agent shouts "stop", gets the piece, and exits.
- The last agent gets the remaining piece.

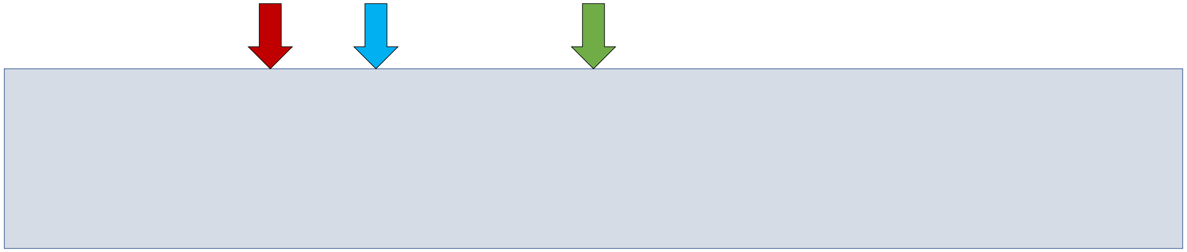
Dubins-Spanier



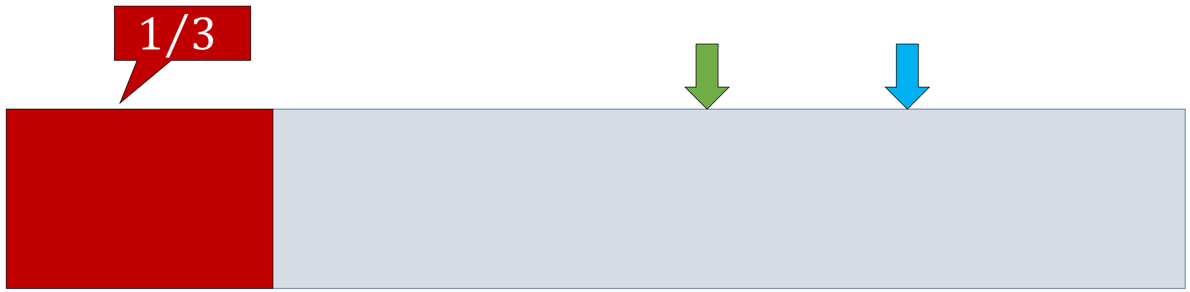
Dubins-Spanier

- Moving a knife continuously is not really needed.
- At each stage, we can ask each remaining agent a cut query to mark his $1/n$ point in the remaining cake.
- Move the knife to the leftmost mark.

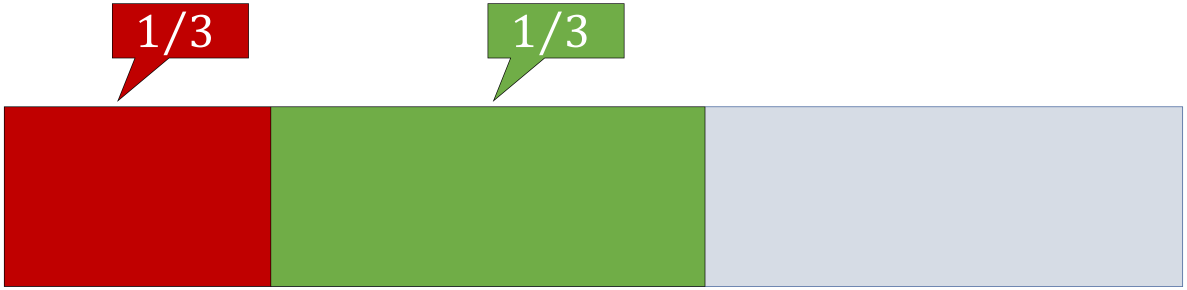
Dubins-Spanier



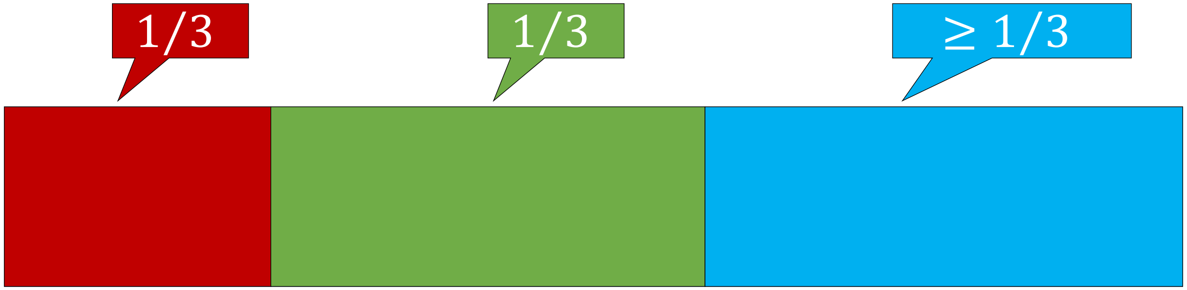
Dubins-Spanier



Dubins-Spanier



Dubins-Spanier



DUBINS-SPANIER

- Question: What is the complexity of the Dubins-Spanier protocol in the Robertson-Webb model?

1. $\Theta(n)$
2. $\Theta(n \log n)$
3. $\Theta(n^2)$
4. $\Theta(n^2 \log n)$

EVEN-PAZ

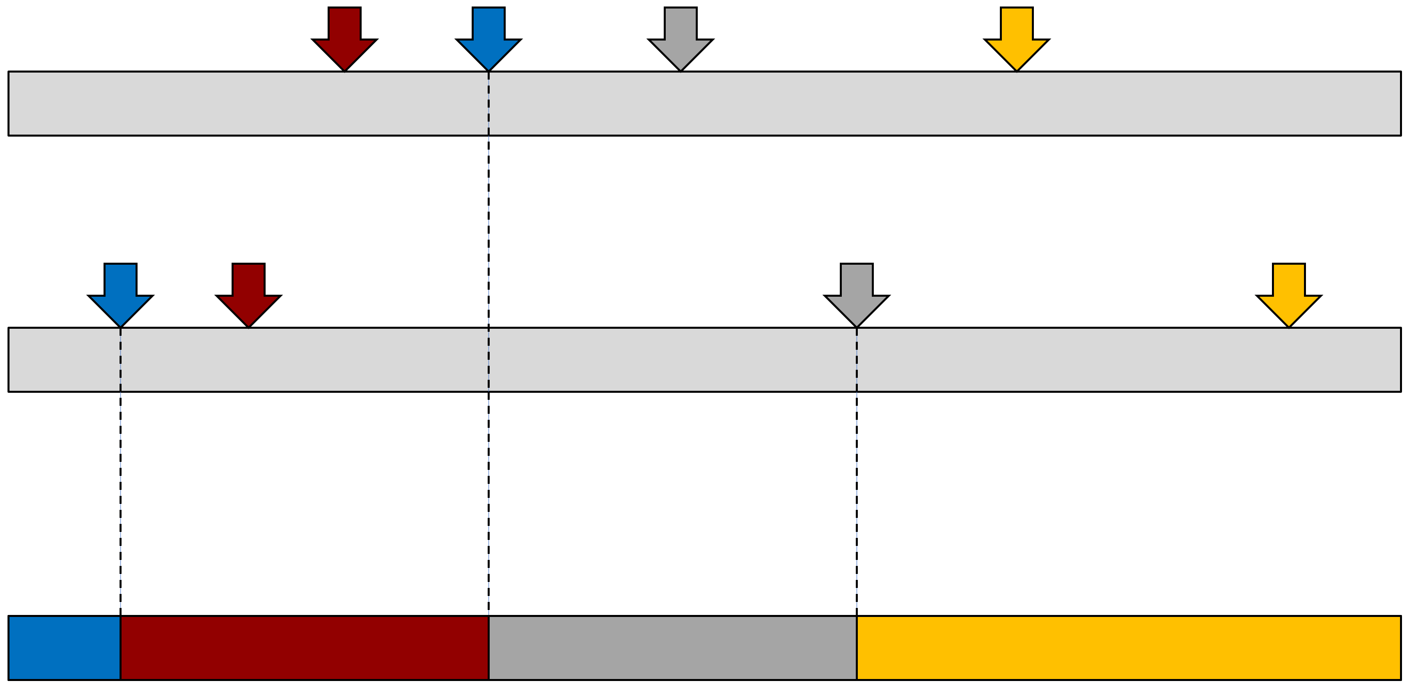
- Input: Interval $[x, y]$, number of agents n
  - Assume $n = 2^k$ for some k

- If $n = 1$, give $[x, y]$ to the single agent.
- Otherwise, let each agent i mark z_i s.t.

$$V_i([x, z_i]) = \frac{1}{2} V_i([x, y])$$

- Let z^* be the $n/2$-th mark from the left.
- Recurse on $[x, z^*]$ with the left $n/2$ agents and on $[z^*, y]$ with the right $n/2$ agents.

Even-Paz



EVEN-PAZ

- Theorem: EVEN-PAZ returns a Prop allocation.
- Proof:
  - Inductive proof. We want to prove that if agent i is allocated piece A_i when $[x, y]$ is divided between n agents, $V_i(A_i) \geq (1/n)V_i([x, y])$
    - Then Prop follows because initially $V_i([x, y]) = V_i([0,1]) = 1$
  - Base case: $n = 1$ is trivial.
  - Suppose it holds for $n = 2^{k-1}$. We prove for $n = 2^k$.
  - Take the 2^{k-1} left agents.
    - Every left agent i has $V_i([x, z^*]) \geq (1/2)\ V_i([x, y])$
    - If it gets A_i, by induction, $V_i(A_i) \geq \frac{1}{2^{k-1}}\ V_i([x, z^*]) \geq \frac{1}{2^k}\ V_i([x, y])$

Even-Paz

- Question: What is the complexity of the Even-Paz protocol in the Robertson-Webb model?

1. $\Theta(n)$
2. $\Theta(n \log n)$
3. $\Theta(n^2)$
4. $\Theta(n^2 \log n)$

Complexity of Proportionality

- Theorem [Edmonds and Pruhs, 2006]: Any proportional protocol needs $\Omega(n \log n)$ operations in the Robertson-Webb model.

- Thus, the EVEN-PAZ protocol is (asymptotically) provably optimal!

Envy-Freeness?

- "I suppose you are also going to give such cute algorithms for finding envy-free allocations?"

- Bad luck. For n-agent EF cake-cutting:
  - [Brams and Taylor, 1995] gave an unbounded EF protocol.
  - [Procaccia 2009] proved $\Omega(n^2)$ lower bound for EF.
  - In 2016, the long-standing major open question of "bounded EF protocol" was resolved!
  - [Aziz and Mackenzie, 2016]: $O(n^{n^{n^{n^{n^n}}}})$ protocol!
    - Not a typo!

Perfect Partition

- Definition:
  - $(B_1, \ldots, B_n)$ is a perfect partition if $V_i(B_j) = {}^1/_n$ for all $i, j \in [n]$
  - Implies envy-freeness (and thus proportionality) and equitability
- Theorem [Lyapunov '40]:
  - There always exists a "perfect partition" of the cake.
- Theorem [Alon '87]:
  - There exists a perfect partition with at most $n(n-1)$ cuts
- Unfortunately, computing a perfect partition needs an unbounded number of RW queries

Pareto Optimality (PO)

- Definition
  - Allocation $A = (A_1, \ldots, A_n)$ is Pareto optimal (PO) if there is no alternative allocation $B = (B_1, \ldots, B_n)$ such that
  1. Every agent is at least as happy: $V_i(B_i) \geq V_i(A_i), \forall i \in N$
  2. Some agent is strictly happier: $V_i(B_i) > V_i(A_i), \exists i \in N$

- Q: Is it PO to give the entire cake to agent 1?
  - A: Not necessarily. But yes, if agent 1 values every part of the cake positively.
  - But a “sequential dictatorship” is always Pareto optimal
    - Let agent 1 take whatever she values positively
    - From the rest, let agent 2 take whatever she values positively
    - And so on…

PO + EF

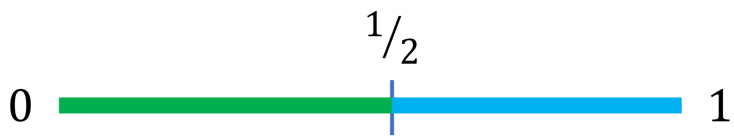
- Theorem [Weller '85]:
  - There always exists an allocation of the cake that is both envy-free and Pareto optimal.
  - Nonconstructive proof via Kakutani's fixed point theorem

- A constructive proof due to [Ebadian, Freeman, Shah, '24]
- Maximum Nash welfare (MNW) allocation
  - A is an MNW allocation if it maximizes the Nash welfare $\prod_{i \in N} V_i(A_i)$ (named after John Nash) across all allocations

MNW Allocation

- Example:
  - Green agent has value 1 distributed over $[0, {}^2/_3]$
  - Blue agent has value 1 distributed over $[0,1]$
  - Without loss of generality (why?) suppose:
    - Green agent gets x fraction of $[0, {}^2/_3]$
    - Blue agent gets the remaining $1 - x$ fraction of $[0, {}^2/_3]$ AND all of $[{}^2/_3, 1]$.
  - Green's utility = x, blue's utility = $(1 - \text{x}) \cdot \frac{2}{3} + \frac{1}{3} = \frac{3-2x}{3}$
  - Maximize: $x \cdot \frac{3-2x}{3} \Rightarrow x = {}^3/_4$ (${}^3/_4$ fraction of ${}^2/_3$ is ${}^1/_2$).

Allocation



Green has utility $\frac{3}{4}$

Blue has utility $\frac{1}{2}$

Maximum Nash Welfare

- Lemma [Segal-Halevi & Sziklai, '19]:
  An MNW allocation of the cake exists.

- Proof:
  - Let $U = \{(v_1(A_1), \dots, v_n(A_n)) : A \text{ is an allocation of the cake}\}$ be the set of feasible utility vectors
  - Dubins and Spanier (1961): U is compact and convex
  - Weierstrass' Extreme Value Theorem: Any continuous function attains a maximum over a compact space.
  - Hence, there exists $u^* \in U$ that is in $\text{argmax}_{u \in U} \prod_i u_i$
  - Any allocation A^* that induced u^* is an MNW allocation

Maximum Nash Welfare

- Theorem [Segal-Halevi & Sziklai, '19; Ebadian, Freeman, Shah, '24]: Any MNW allocation of the cake is EF+PO.

- Proof:
  - Let A be an MNW allocation
  - Note that $\prod_i v_i(A_i) > 0$ (because even a proportional allocation achieves a positive Nash welfare), so $v_i(A_i) > 0, \forall i$
  - PO follows from the fact that any Pareto improvement would have a strictly higher Nash welfare
  - Suppose for contradiction that A is not EF and $v_i(A_j) > v_i(A_i)$

Maximum Nash Welfare

- Proof (continued):
  - Consider the allocation A' obtained by reallocating A_j to agent i
    - $A'_i = A_i \cup A_j$, $A'_j = \emptyset$, $A'_k = A_k \; \forall k \neq i, j$
    - $v_i(A'_i) > 2 \cdot v_i(A_i)$, $v_k(A'_k) = v_k(A_k) \; \forall k \neq i, j$
    - Let u and u' be the utility vectors induced by A and A'

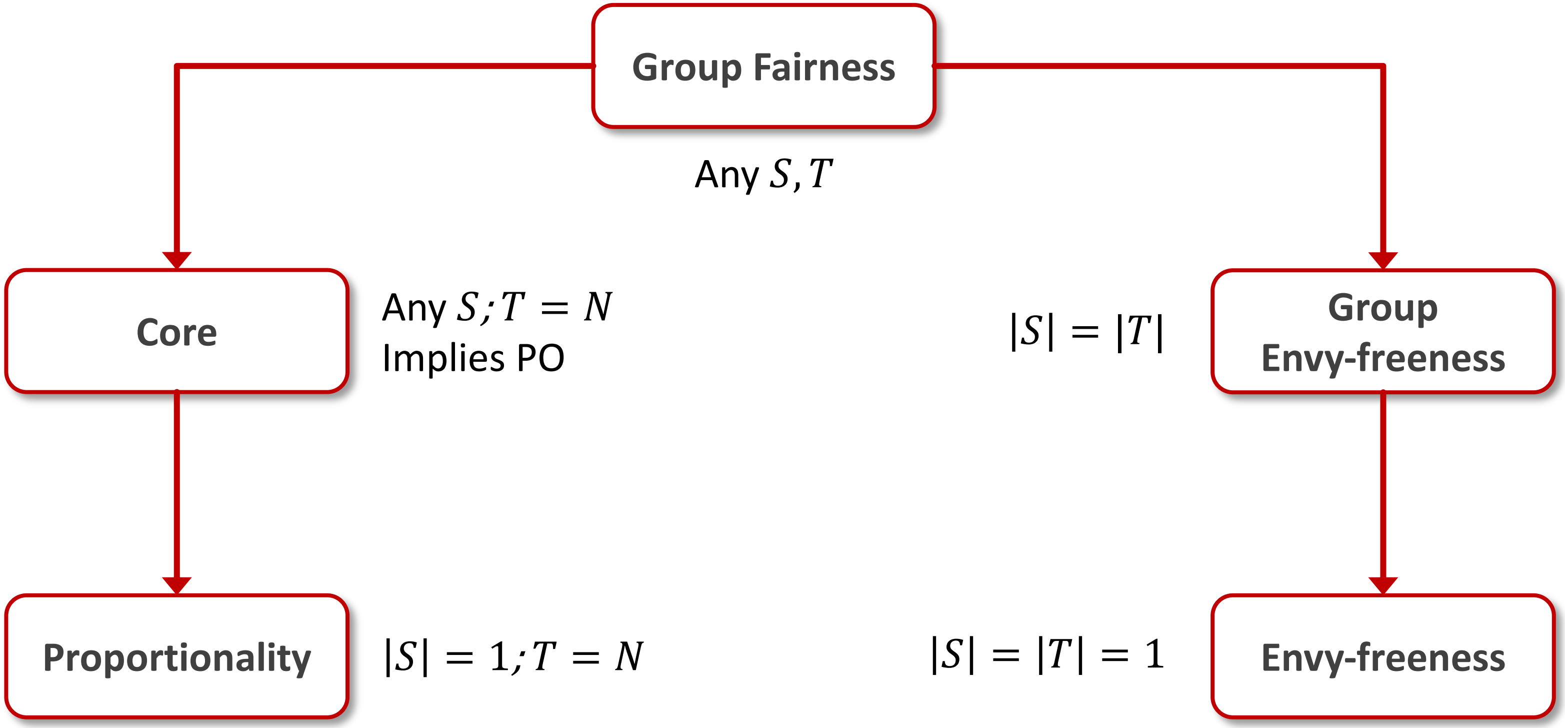
  - For $\lambda \in [0,1]$, let $u^\wedge\lambda = \lambda \cdot u + (1-\lambda) \cdot u'$ and $f(\lambda) = \sum_i \log u_i^\lambda$
  - Due to convexity of U, $u^\lambda \in U \; \forall \lambda \in [0,1]$
  - For the contradiction, suffices to prove that $\exists \lambda \in [0,1]$: $f(\lambda) > f(1)$
  - Since $f(\lambda)$ is differentiable in λ, enough to prove that $f'(1) < 0$ (proof on board). ■

Group Fairness

- An allocation A is called group fair (GF) if…
  - there are no subsets of agents $S, T \subseteq N$ and reallocation $\cup_{i \in T} A_i \Rightarrow (B_i : i \in S)$ of the collective allocation of T to agents in S such that
  - $\frac{|S|}{|T|} \cdot V_i(B_i) \geq V_i(A_i)$ for all $i \in S$ and at least one inequality is strict

- Theorem [Conitzer et al. '19; Freeman et al. '20]
  - For cake-cutting, any MNW allocation satisfies group fairness.
  - Among allocation rules satisfying a mild additional axiom, it is the only rule that does so.

Group Fairness

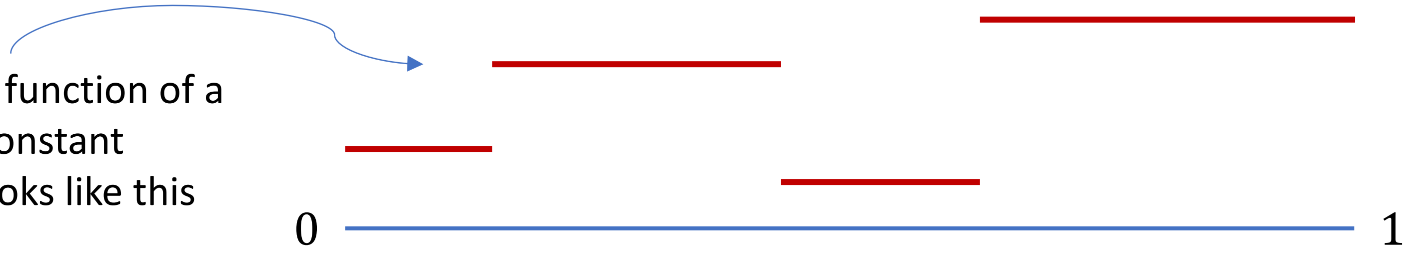


Problem with Nash Solution

- Computing any Pareto optimal allocation already requires an unbounded number of queries

- Theorem [Aziz & Ye '14]:
  - For *piecewise constant* valuations, the Nash-optimal solution can be computed in polynomial time.

The density function of a piecewise constant valuation looks like this



Homogeneous Divisible Goods

- Suppose there are m homogeneous divisible goods
  - Each good can be divided fractionally between the agents
- Let $x_{i,g}$ = fraction of good g that agent i gets
  - Homogeneous = agent doesn't care which "part"
- Special case of cake-cutting
  - Line up the goods on $[0,1]$

Homogeneous Divisible Goods

- MNW solution:

  Maximize $\sum_i \log U_i$

  $U_i = \Sigma_g \, x_{i,g} * v_{i,g} \qquad \forall i$

  $\Sigma_i \, x_{i,g} = 1 \qquad \forall g$

  $x_{i,g} \in [0,1] \qquad \forall i, g$

- This is known as the Gale-Eisenberg convex program
  - Can be solved *exactly* in strongly polynomial time [Orlin '10]