CSC2421 Spring'24
Assignment 1
Due Date: Mar 10, 2024

Notes

1. *Citation policy:*
   - It is certainly preferable for you to solve the questions without consulting a peer, an AI, or an online source. However, if you do consult and obtain useful insights, you must cite the name of the peer/AI or the link of the source you referred.
   - Further, you should write the solution in your own words. The best way to do so is to not take any notes during discussions, spend at least a few hours playing a video game or reading a novel, and then construct the solution on your own.
2. *No garbage policy:* Leaving an answer blank will get you 20% of the points (this also applies to a subproblem). This does not apply to any bonus questions.
3. Typed assignments are highly preferred (LaTeX or Word), especially if your handwriting is possibly illegible or if you do not have access to a good quality scanner. Please submit *a single PDF* on MarkUs.

Total Marks: 100 across 4 questions.

Q1 [25 Points] Is Fairness Restrictive?

Consider the cake cutting problem in which n agents have valuation functions $V_1, \ldots, V_n$ satisfying the standard additivity, normalization, and divisibility assumptions we stated in class. Denote the *social welfare* of an allocation $\mathbf{A}$ by $\mathrm{sw}(\mathbf{A}) = \sum_{i=1}^{n} V_i(A_i)$.

In the questions below, we are interested in measuring how restrictive the notion of proportionality is. Specifically, we would like to measure the worst-case multiplicative loss in social welfare that one *must* incur when imposing proportionality. To do so, we compare the maximum social welfare we can achieve *without* requiring proportionality to the maximum social welfare we can achieve *subject to* proportionality.

(a) [15 Points] Show that for all possible valuations $V_1, \ldots, V_n$,

$$\frac{\max\{\mathrm{sw}(\mathbf{A}) : \mathbf{A} \text{ is an allocation of the cake}\}}{\max\{\mathrm{sw}(\mathbf{A}) : \mathbf{A} \text{ is a } \textit{proportional} \text{ allocation of the cake}\}} = O(\sqrt{n}).$$

[Hint: Consider an allocation $\mathbf{A}^*$ that maximizes social welfare. Let L be the set of agents who have value at least $1/\sqrt{n}$ for their piece of the cake under $\mathbf{A}^*$. Consider two cases: $|L| < \sqrt{n}$ and $|L| \geq \sqrt{n}$. The former case is easy. In the latter case, shuffle the allocations of the agents in $\mathbf{A}^*$ to generate a proportional allocation $\mathbf{A}$ that does not lose too much welfare compared to $\mathbf{A}^*$.]

(b) [10 Points] Give a family of examples of $V_1, \ldots, V_n$ (one example for each value of n) such that

$$\frac{\max\{\mathrm{sw}(\mathbf{A}) : \mathbf{A} \text{ is an allocation of the cake}\}}{\max\{\mathrm{sw}(\mathbf{A}) : \mathbf{A} \text{ is a } \textit{proportional} \text{ allocation of the cake}\}} = \Omega(\sqrt{n}).$$

Q2 [25 Points] Maximin Share

Consider the setting of allocating indivisible goods, where a set of goods M is to be allocated to a set of n agents N with additive valuations $V_1, \dots, V_n$. Recall the definition of maximin share from class. For a subset of goods S, let $T_k(S)$ be the set of all partitions of S into k bundles, and

$$\text{MMS}_i(k, S) = \max_{T \in T_k(S)} \min_{T_j \in T} v_i(T_j)$$

be the maximum value placed by agent i on the worst bundle across all such partitions. We say that an allocation A is α-MMS if $V_i(A_i) \geq \alpha \cdot \text{MMS}_i(n, M)$ for all i. In this question, you will derive a simple $1/2$-MMS approximation.

Define $\text{PROP}_i(k, S) = \frac{1}{k} \sum_{g \in S} V_i(\{g\})$. Consider the following algorithm.

Algorithm 1: $1/2$-MMS

1 **while** $\exists i, g : V_i(\{g\}) \geq \frac{1}{2} \cdot PROP_i(|N|, M)$ **do** // If i values g a lot
2 $\quad A_i \leftarrow \{g\}$; // Allocate g to i
3 $\quad N \leftarrow N \setminus \{i\}, M \leftarrow M \setminus \{g\}$; // Remove g, i forever
4 Run round robin to allocate the remaining goods in M to the remaining agents in N, and store the results in A;
5 **return** A;

(a) [10 Points] Prove that $\text{MMS}_i(n-1, M \setminus \{g\}) \geq \text{MMS}_i(n, M)$. That is, the MMS value of an agent can only go up if one other agent and one good are removed from consideration.

(b) [5 Points] Argue that $\text{MMS}_i(k, S) \leq \text{PROP}_i(k, S)$. Use that to deduce that the $1/2$-MMS guarantee is satisfied for every agent allocated to (and removed) by the while loop of the algorithm.

(c) [10 Points] Assume that the round robin procedure, when used to find an allocation A of a set of goods M to a set of agents N, satisfies the following property (it is implied by EF1): $V_i(A_i) \geq \text{PROP}_i(|N|, M) - \max_{g \in M} V_i(\{g\})$. Use it to prove that the $1/2$-MMS guarantee is also satisfied for all the agents allocated to by Step 4 of the algorithm. (Hint: Use the fact that each such agent must not value every remaining good too highly.)

Q3 [25 Points] Stronger Justified Representation Recall the EJR guarantee for approval-based committee selection from class. A committee W of size k satisfies EJR if

- for all $\ell \in \{1, \dots, k\}$ and groups of voters $S \subseteq N$ that are...
- $|S| \geq \ell \cdot n/k$ (large) and $|\cap_{i \in S} A_i| \geq \ell$ (cohesive)...
- $u_i(W) = |A_i \cap W| \geq \ell$ for at least one $i \in S$.

One of the students asked why we should only demand at least one member to have utility at least ℓ and not for each member to have utility at least ℓ, which would be a stronger guarantee. In this question, you will show that this stronger guarantee cannot always be provided.

Consider an election with four candidates $\{a, b, c, d\}$ and 12 voters with approval sets $(\{a, b\}, \{b\}, \{b\}, \{b, c\}, \{c\}, \{c\}, \{c, d\}, \{d\}, \{d\}, \{d, a\}, \{a\}, \{a\})$. Notice the cyclic nature of this list. Argue that no committee of size $k = 3$ will satisfy the strong notion suggested above. (Hint: For each candidate, find a group of voters which would require that candidate to be part of the committee.)

Q4 [25 Points] Fun with Deferred Acceptance

Consider the Deferred Acceptance algorithm to find a stable matching between n men and n women where each participant has a strict ranking over participants of the opposite gender.

(a) [15 Points] Consider the following preferences for 4 men (M1 through M4) and 4 women (W1 through W4). Each row gives the preference of one individual, and the preference decreases from left (most preferred) to right (least preferred).

Men's Preferences

M1	W2	W4	W1	W3
M2	W3	W1	W4	W2
M3	W2	W3	W1	W4
M4	W4	W1	W3	W2

Women's Preferences

W1	M2	M1	M4	M3
W2	M4	M3	M1	M2
W3	M1	M4	M3	M2
W4	M2	M1	M4	M3

Run men-proposing deferred acceptance (MPDA) and women-proposing deferred acceptance (WPDA) on this instance. For each algorithm, describe each iteration: who proposes to whom in that iteration, and who is engaged to whom at the end of the iteration.

(b) [10 Points] Suppose there are k "good" men and k "good" women such that in the preference ranking of each woman (resp. man), the top k men (resp. women) are precisely the k good men (resp. women) in some order. That is, every participant prefers the k good participants of the opposite gender to the other participants of the opposite gender. Show that in any stable matching, the k good men must be matched to the k good women.