CSC2421 Spring'24
Assignment 1 Solutions

Q1 [25 Points] Is Fairness Restrictive?

Consider the cake cutting problem in which n agents have valuation functions $V_1, \dots, V_n$ satisfying the standard additivity, normalization, and divisibility assumptions we stated in class. Denote the *social welfare* of an allocation $\mathbf{A}$ by $\text{sw}(\mathbf{A}) = \sum_{i=1}^{n} V_i(A_i)$.

In the questions below, we are interested in measuring how restrictive the notion of proportionality is. Specifically, we would like to measure the worst-case multiplicative loss in social welfare that one *must* incur when imposing proportionality. To do so, we compare the maximum social welfare we can achieve *without* requiring proportionality to the maximum social welfare we can achieve *subject to* proportionality.

(a) [15 Points] Show that for all possible valuations $V_1, \dots, V_n$,

$$\frac{\max\{\text{sw}(\mathbf{A}) : \mathbf{A} \text{ is an allocation of the cake}\}}{\max\{\text{sw}(\mathbf{A}) : \mathbf{A} \text{ is a } \textit{proportional} \text{ allocation of the cake}\}} = O(\sqrt{n}).$$

[Hint: Consider an allocation $\mathbf{A}^*$ that maximizes social welfare. Let L be the set of agents who have value at least $1/\sqrt{n}$ for their piece of the cake under $\mathbf{A}^*$. Consider two cases: $|L| < \sqrt{n}$ and $|L| \geq \sqrt{n}$. The former case is easy. In the latter case, shuffle the allocations of the agents in $\mathbf{A}^*$ to generate a proportional allocation $\mathbf{A}$ that does not lose too much welfare compared to $\mathbf{A}^*$.]

(b) [10 Points] Give a family of examples of $V_1, \dots, V_n$ (one example for each value of n) such that

$$\frac{\max\{\text{sw}(\mathbf{A}) : \mathbf{A} \text{ is an allocation of the cake}\}}{\max\{\text{sw}(\mathbf{A}) : \mathbf{A} \text{ is a } \textit{proportional} \text{ allocation of the cake}\}} = \Omega(\sqrt{n}).$$

Solution to Q1

This is an adaptation of the proof of Caragiannis et al. [1] due to Procaccia [2].

(a) Let $V_1, \dots, V_n$ denote the valuations of the players, and let $\mathbf{A}^* \in \arg\max_{\mathbf{A}} \text{sw}(\mathbf{A})$ be an optimal allocation. Let $L = \{i \in N : V_i(A_i^*) \geq 1/\sqrt{n}\}$ be the set of "large" players, and $S = N \setminus L$ be the set of "small" players. Consider two cases.

Case 1: $|L| < \sqrt{n}$. This is the easy case. Note that $\text{sw}(\mathbf{A}^*) \leq |L| \cdot 1 + |S| \cdot 1/\sqrt{n} < 2/\sqrt{n}$. In contrast, *any* proportional allocation $\mathbf{A}$ satisfies $\text{sw}(\mathbf{A}) \geq n \cdot 1/n = 1$, yielding an approximation of $O(\sqrt{n})$.

Case 2: $|L| \geq \sqrt{n}$. This is the more difficult case. We will construct *a specific* proportional allocation $\mathbf{A}$ such that $\text{sw}(\mathbf{A}) \geq \text{sw}(\mathbf{A}^*)/\sqrt{n}$. Note that $|L| \geq \sqrt{n}$ implies $|S| \leq n - \sqrt{n}$. Define an allocation $\mathbf{A}$ by reallocating pieces in $\mathbf{A}^*$ to the players as follows.

For each $i \in S$, allocate A_i^* among players in S in a proportional manner, so that for each $j \in S$, we have $V_j(A_j \cap A_i^*) \geq V_j(A_i^*)/|S|$.

For each $i \in L$, allocate A_i^* among players in $\{i\} \cup S$ so that

$$V_i(A_i \cap A_i^*) \geq \sqrt{n} \cdot \frac{V_i(A_i^*)}{\sqrt{n} + |S|},$$

and for every $j \in S$,

$$V_j(A_j \cap A_i^*) \geq \frac{V_j(A_i^*)}{\sqrt{n} + |S|}.$$

One way to achieve this is to create $\sqrt{n}$ copies of player i with identical preferences, and dividing A_i^* proportionally among the $\sqrt{n}$ copies as well as players in S (i.e., among a total of $\sqrt{n} + |S|$ players), and giving player i the union of what the copies receive.

We now show that $\mathbf{A}$ is proportional, and $\mathrm{sw}(\mathbf{A}) \geq \mathrm{sw}(\mathbf{A}^*)/\sqrt{n}$. For $i \in L$, we have

$$V_i(A_i) \geq \sqrt{n} \cdot \frac{V_i(A_i^*)}{\sqrt{n} + |S|} \geq \frac{1}{\sqrt{n} + |S|} \geq \frac{1}{n},$$

and for $i \in S$, we have

$$V_i(A_i) \geq \sum_{j \in L} \frac{V_i(A_j^*)}{\sqrt{n} + |S|} + \sum_{j \in S} \frac{V_i(A_j^*)}{|S|} \geq \frac{\sum_{j \in N} V_i(A_j^*)}{n} = \frac{1}{n}.$$

Hence, $\mathbf{A}$ is a proportional allocation. We next show that $V_i(A_i) \geq V_i(A_i^*)/\sqrt{n}$, which would yield the desired approximation ratio of $\sqrt{n}$. For $i \in S$, this holds because $V_i(A_i^*) \leq 1/\sqrt{n}$ and $V_i(A_i) \geq 1/n$. For $i \in L$, it follows because

$$V_i(A_i) \geq \sqrt{n} \cdot \frac{V_i(A_i^*)}{\sqrt{n} + |S|} \geq \sqrt{n} \cdot \frac{V_i(A_i^*)}{n} = \frac{V_i(A_i^*)}{\sqrt{n}}.$$

(b) For the lower bound, consider the following valuations for any given n.

The set $L \subseteq N$ contains exactly $\sqrt{n}$ players. Each player $i \in L$ uniformly desires a single interval of length $1/\sqrt{n}$, and the desired intervals of any two players $i, j \in L$ are disjoint. The set of players $S = N \setminus L$ contains $n - \sqrt{n}$ players who desire the entire cake uniformly.

The optimal allocation $\mathbf{A}^*$ gives the players in L their desired intervals, achieving $\mathrm{sw}(\mathbf{A}^*) = \sqrt{n}$. In contrast, any proportional allocation $\mathbf{A}$ must give each player in S an interval of length at least $1/n$, leaving only $1/\sqrt{n}$ length of the cake for players in L. Because players in L have value density at most $\sqrt{n}$ at any point, it must hold that $\sum_{i \in L} V_i(A_i) \leq \sqrt{n} \cdot 1/\sqrt{n} = 1$, while $\sum_{i \in S} V_i(A_i) \leq 1$. Hence, $\mathrm{sw}(\mathbf{A}) \leq 2$, yielding an approximation ratio of $\Omega(\sqrt{n})$.

Q2 [25 Points] Maximin Share

Consider the setting of allocating indivisible goods, where a set of goods M is to be allocated to a set of n agents N with additive valuations $V_1, \ldots, V_n$. Recall the definition of maximin share from class. For a subset of goods S, let $T_k(S)$ be the set of all partitions of S into k bundles, and

$$\text{MMS}_i(k, S) = \max_{T \in T_k(S)} \min_{T_j \in T} v_i(T_j)$$

be the maximum value placed by agent i on the worst bundle across all such partitions. We say that an allocation A is α-MMS if $V_i(A_i) \geq \alpha \cdot \text{MMS}_i(n, M)$ for all i. In this question, you will derive a simple $1/2$-MMS approximation.

Define $\text{PROP}_i(k, S) = \frac{1}{k} \sum_{g \in S} V_i(\{g\})$. Consider the following algorithm.

Algorithm 1: $1/2$-MMS

1 **while** $\exists i, g : V_i(\{g\}) \geq \frac{1}{2} \cdot PROP_i(|N|, M)$ **do** // If i values g a lot
2 $\quad A_i \leftarrow \{g\}$; // Allocate g to i
3 $\quad N \leftarrow N \setminus \{i\}, M \leftarrow M \setminus \{g\}$; // Remove g, i forever
4 Run round robin to allocate the remaining goods in M to the remaining agents in N, and store the results in A;
5 **return** A;

(a) [10 Points] Prove that $\text{MMS}_i(n-1, M \setminus \{g\}) \geq \text{MMS}_i(n, M)$. That is, the MMS value of an agent can only go up if one other agent and one good are removed from consideration.

(b) [5 Points] Argue that $\text{MMS}_i(k, S) \leq \text{PROP}_i(k, S)$. Use that to deduce that the $1/2$-MMS guarantee is satisfied for every agent allocated to (and removed) by the while loop of the algorithm.

(c) [10 Points] Assume that the round robin procedure, when used to find an allocation A of a set of goods M to a set of agents N, satisfies the following property (it is implied by EF1): $V_i(A_i) \geq \text{PROP}_i(|N|, M) - \max_{g \in M} V_i(\{g\})$. Use it to prove that the $1/2$-MMS guarantee is also satisfied for all the agents allocated to by Step 4 of the algorithm. (Hint: Use the fact that each such agent must not value every remaining good too highly.)

Solution to Q2

(a) Consider a partition T of M into n bundles such that $\text{MMS}_i(n, M) = \min_{T_j \in T} v_i(T_j)$ (this is a partition that maximizes the value of the minimum bundle, thus achieving the MMS value of the agent). We derive a partition T' of $M \setminus \{g\}$ into $n-1$ bundles such that $\text{MMS}_i(n, M) \leq \min_{T'_j \in T'} v_i(T'_j) \leq \text{MMS}_i(n-1, M \setminus \{g\})$, where the last inequality is due to the definition of $\text{MMS}_i(n-1, M \setminus \{g\})$. This would prove the required relation.

Note that good g must be part of some bundle $T_k \in T$. Create T' by starting with the $n-1$ bundles of T other than T_k, and distributing the goods in $T_k \setminus \{g\}$ among these $n-1$ bundles arbitrarily. Note that $\min_{T'_j \in T'} v_i(T'_j) \geq \min_{T_j \in T \setminus \{T_k\}} v_i(T_j) \geq \min_{T_j \in T} v_i(T_j)$, where the first inequality holds because each bundle in T' is a superset of the corresponding bundle from $T \setminus \{T_k\}$. This completes the proof.

(b) Consider any partition $T \in T_k(S)$. Note that

$$\min_{T_j \in T} v_i(T_j) \leq \frac{1}{k} \sum_{T_j \in T} v_i(T_j) = \frac{1}{k} \sum_{g \in S} v_i(\{g\}) = \text{PROP}_i(k, S),$$

where the second transition is due to additivity of valuations. Since this holds for all $T \in T_k(S)$, we have that $\text{MMS}_i(k, S) = \max_{T \in T_k(S)} \min_{T_j \in T} v_i(T_j) \leq \text{PROP}_i(k, S)$.

Consider any agent i who is allocated a good g at some point by the while loop. Let N' be the set of agents remaining and M' be the set of goods remaining right before agent i was allocated good g. Then, by the condition of the while loop, we must've had $v_i(\{g\}) \geq \frac{1}{2} \cdot \text{PROP}_i(|N'|, M') \geq \frac{1}{2} \cdot \text{MMS}_i(|N'|, M')$, where the last inequality is due to what we just proved. Further, since every iteration of the while loop removes one agent and one good from the system, due to part (a) the MMS value of every remaining agent for the remaining set of goods (weakly) goes up. Hence, $\text{MMS}_i(|N'|, M') \geq \text{MMS}_i(n, M)$. Together, we have that $v_i(A_i) = v_i(\{g\}) \geq \frac{1}{2} \cdot \text{MMS}_i(n, M)$, as needed.

(c) Let N' and M' be the sets of agents and goods remaining when the round robin procedure is called. Consider any agent $i \in N'$. Because this agent was not considered by the while loop, it must be the case that $v_i(\{g\}) < \frac{1}{2} \cdot \text{PROP}_i(|N'|, M')$ for all $g \in M'$. Hence, $\max_{g \in M'} v_i(\{g\}) < \frac{1}{2} \cdot \text{PROP}_i(|N'|, M')$. Using the property of round robin given to us, we have that

$$v_i(A_i) \geq \text{PROP}_i(|N'|, M') - \max_{g \in M'} v_i(\{g\}) > \frac{1}{2} \cdot \text{PROP}_i(|N'|, M') \geq \frac{1}{2} \cdot \text{MMS}_i(|N'|, M') \geq \frac{1}{2} \cdot \text{MMS}_i(n, M),$$

where the penultimate inequality is due to part (b), and the last inequality is due to the same reasoning as provided in part (b) (MMS values go up for all remaining agents due to the while loop removing one agent and one good at a time). This completes the proof.

Q3 [25 Points] Stronger Justified Representation

Recall the EJR guarantee for approval-based committee selection from class. A committee W of size k satisfies EJR if

- for all $\ell \in \{1, \ldots, k\}$ and groups of voters $S \subseteq N$ that are...
- $|S| \geq \ell \cdot n/k$ (large) and $| \cap_{i \in S} A_i| \geq \ell$ (cohesive)...
- $u_i(W) = |A_i \cap W| \geq \ell$ for at least one $i \in S$.

One of the students asked why we should only demand at least one member to have utility at least ℓ and not for each member to have utility at least ℓ, which would be a stronger guarantee. In this question, you will show that this stronger guarantee cannot always be provided.

Consider an election with four candidates $\{a, b, c, d\}$ and 12 voters with approval sets $(\{a, b\}, \{b\}, \{b\}, \{b, c\}, \{c\}, \{c\}, \{c, d\}, \{d\}, \{d\}, \{d, a\}, \{a\}, \{a\})$. Notice the cyclic nature of this list. Argue that no committee of size $k = 3$ will satisfy the strong notion suggested above. (Hint: For each candidate, find a group of voters which would require that candidate to be part of the committee.)

Solution to Q3

Any committee of size $k = 3$ would leave one candidate unselected. Consider any arbitrary committee. Due to the symmetry of the profile, assume, without loss of generality, that the candidate not selected is d. Then, consider the group of voters S whose votes are $\{c, d\}, \{d\}, \{d\}, \{d, a\}$.

Note that the group satisfies the stronger JR requirement for $\ell = 1$ because $|S| = 4 = 1 \cdot 12/3$ (large) and $\cap_{i \in S} A_i = \{d\}$ (cohesive). Hence, stronger JR would require that $|A_i \cap W| \geq 1$ for all $i \in S$. However, this is not the case for the two voters whose votes are $\{d\}$. Hence, the arbitrarily selected committee does not satisfy stronger JR, i.e., no committee satisfies stronger JR on this profile.

Q4 [25 Points] Fun with Deferred Acceptance

Consider the Deferred Acceptance algorithm to find a stable matching between n men and n women where each participant has a strict ranking over participants of the opposite gender.

(a) [15 Points] Consider the following preferences for 4 men (M1 through M4) and 4 women (W1 through W4). Each row gives the preference of one individual, and the preference decreases from left (most preferred) to right (least preferred).

Men's Preferences

M1	W2	W4	W1	W3
M2	W3	W1	W4	W2
M3	W2	W3	W1	W4
M4	W4	W1	W3	W2

Women's Preferences

W1	M2	M1	M4	M3
W2	M4	M3	M1	M2
W3	M1	M4	M3	M2
W4	M2	M1	M4	M3

Run men-proposing deferred acceptance (MPDA) and women-proposing deferred acceptance (WPDA) on this instance. For each algorithm, describe each iteration: who proposes to whom in that iteration, and who is engaged to whom at the end of the iteration.

(b) [10 Points] Suppose there are k "good" men and k "good" women such that in the preference ranking of each woman (resp. man), the top k men (resp. women) are precisely the k good men (resp. women) in some order. That is, every participant prefers the k good participants of the opposite gender to the other participants of the opposite gender. Show that in any stable matching, the k good men must be matched to the k good women.

Solution to Q4

(a) MPDA:

Iteration	Proposal	Engagements
1	M1 to W2	M1W2
2	M2 to W3	M1W2, M2W3
3	M3 to W2	M2W3, M3W2
4	M4 to W4	M2W3, M3W2, M4W4
5	M1 to W4	M2W3, M3W2, M1W4
6	M4 to W1	M2W3, M3W2, M1W4, M4W1

WPDA:

Iteration	Proposal	Engagements
1	W1 to M2	W1M2
2	W2 to M4	W1M2, W2M4
3	W3 to M1	W1M2, W2M4, W3M1
4	W4 to M2	W1M2, W2M4, W3M1
5	W4 to M1	W1M2, W2M4, W4M1
6	W3 to M4	W1M2, W4M1, W3M4
7	W2 to M3	W1M2, W4M1, W3M4, W2M3

(b) For contradiction, suppose w.l.o.g. that there is a “good” man M matched to a “not-good” woman W. Then, there are at most $k - 1$ good man - good woman matches. Hence, there is at least one good woman W′ matched to a not-good man M′. This matching is unstable because W′ prefers M to M′ and M also prefers W′ to W. Thus, in any stable matching, all k good men must be matched to the k good women.

(c) Recall that MPDA returns the stable matching where all men are matched to their best valid partners and all women to their worst valid partners, while WPDA returns the stable matching where all men are matched to their worst valid partners and all women to their best valid partners.

If these are the same stable matching, then for every man and woman, their best valid partner is the same as their worst valid partner, implying that they have a unique valid partner. This in turn implies that there is a unique stable matching in that problem.

References

[1] I. Caragiannis, C. Kaklamanis, P. Kanellopoulos, and M. Kyropoulou. The efficiency of fair division. In *Proceedings of the 5th Conference on Web and and Network Economics (WINE)*, pages 475–482, 2009.

[2] A. D. Procaccia. Cake cutting algorithms. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 13. Cambridge University Press.